High School Mathematics Contest

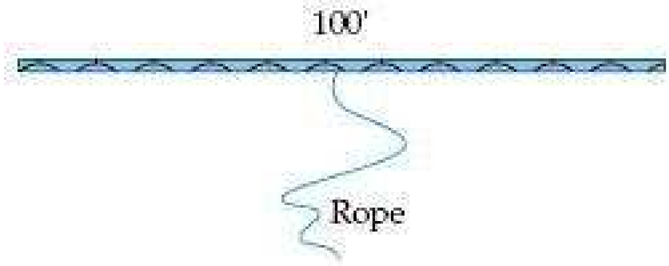
Elon University Mathematics Department

Saturday, April 2, 2011

1. A group of boys and girls are in a room. 15 girls leave the room. Then there are twice as many boys in the room as girls. After this 45 boys leave the room. Then the number of girls in the room is five times the number of boys. How many girls were in the room before anyone left the room?

   (a) 29.
   (b) 40.
   (c) 50.
   (d) 73.
   (e) none of the above.

2. Suppose a goat was tethered at the middle of a 100-foot-long fence and the length of the rope was 29 feet, as shown in the diagram below. Over how much area could the goat graze? Use 3.14 for π and round your answer to 2 decimal places.



   (a) 91.06 ft^2.
   (b) 1261.50 ft^2.
   (c) 1320.37 ft^2.
   (d) 2640.74 ft^2.
   (e) 5281.48 ft^2.

3. A square and a circle have equal perimeters. The ratio of the area of the circle to the area of the square is

   (a) $\frac{4}{\pi}$.
   (b) $\frac{\pi}{\sqrt{2}}$.
   (c) $\frac{4}{1}$.
   (d) $\frac{\sqrt{2}}{\pi}$.
   (e) $\frac{\pi}{4}$.

4. Suppose f is a function on the positive integers which takes integer values. Find $f(2011)$ if:
   - $f(2) = 2$
   - $f(mn) = f(m)f(n)$ for all positive integers m and n
   - $f(m) > f(n)$ if $m > n$

   (a) 2011.
   (b) 2012.
   (c) 2048.
   (d) 2^{2011}.
   (e) none of the above.

5. The following statements are true:
   - If a is greater than b, then c is greater than d
   - If c is less than d then e is greater than f

   A valid conclusion is:

   (a) If a is less than b, then e is greater than f.
   (b) If e is greater than f, then a is less than b.
   (c) If e is less than f, then a is greater than b.
   (d) If a is greater than b, then e is less than f.
   (e) none of these.

6. A cylinder has radius r and height h, both measured in centimeters. When measured in cubic centimeters and square centimeters, respectively, the volume and surface area of the cylinder (including top and bottom) are equal. What is $(r - 2)(h - 2)$?

   (a) 3.
   (b) π.
   (c) 4.
   (d) 4π.
   (e) 12π.

7. Suppose that a particular trait of a person (such as eye color or left handedness) is classified on the basis of one pair of genes and suppose d represents a dominant gene and r a recessive gene. Thus a person with dd genes is pure dominance, one with rr genes is pure recessive, and one with rd or dr is hybrid. The pure dominance and the hybrid are alike in appearance. Children receive one gene from each parent, with each gene being equally likely. If, with respect to a particular trait, two hybrid parents have a total of four children, what is the probability that exactly three of the four children have the outward appearance of the dominant gene?

   (a) $\frac{3}{64}$.
   (b) $\frac{27}{256}$.
   (c) $\frac{27}{64}$.
   (d) $\frac{1}{2}$.
   (e) $\frac{3}{4}$.

8. In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that she knows the answer and $1-p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the probability that a student knew the answer to a question, given that she answered it correctly?

   (a) $\dfrac{p}{m}$.
   (b) $\dfrac{p}{p+m(1-p)}$.
   (c) $\dfrac{mp}{1+(m-1)p}$.
   (d) $\dfrac{p}{mp+(1-p)}$.
   (e) $\dfrac{mp}{1+(m+1)p}$.

9. Find the sum of the distinct solutions for $x^5 - 2x^4 + 3x^3 - 4x^2 + 5x - 6 = 0$.

   (a) $3\sqrt{2} - 4$.
   (b) 0.
   (c) 2.
   (d) 6.
   (e) $1 + 5i$.

10. The negation of the statement "all women are honest" is:

    (a) no women are honest.
    (b) all women are dishonest.
    (c) some women are dishonest.
    (d) no women are dishonest.
    (e) some women are honest.

11. Let S be the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$, and let T be the union of the graph of $y = \tan(x)$ where defined for $0 \le x \le 2\pi$ with its vertical asymptotes. How many points are there in $S \cap T$?

    (a) 0.
    (b) 3.
    (c) 4.
    (d) 5.
    (e) 7.

12. Let $\triangle ABC$ be a triangle with the internal angle bisector $\overline{BD}$ and altitude $\overline{BH}$. If angle $\angle ABC = 30$ degrees and $\angle ACB = 2 \times \angle BAC$ then $\angle DBH$ is

    (a) $10°$.
    (b) $15°$.
    (c) $25°$.
    (d) $50°$.
    (e) $80°$.

13. If $4^x - 4^{x-1} = 24$ then $(2x)^x$ equals:

   (a) $\sqrt{5}$.
   (b) $5\sqrt{5}$.
   (c) 25.
   (d) $25\sqrt{5}$.
   (e) 125.

14. What is the value of $\log_{1/9} \sqrt[3]{2187}$?

   (a) $-\frac{7}{6}$.
   (b) -1.
   (c) $-\frac{1}{9}$.
   (d) $\frac{1}{9}$.
   (e) $\frac{7}{6}$.

15. The smallest value of $x^2 + 16x$ for real values of x is

   (a) -256.
   (b) -64.
   (c) -16.
   (d) -0.
   (e) None of these.

16. Let m be an integer selected at random from 1 through 1,000 inclusive. What is the probability (rounded to the nearest percent) that m has more digits when written in base 9 then when written in base 10?

   (a) 29%.
   (b) 30%.
   (c) 31%.
   (d) 32%.
   (e) 33%.

17. How many solutions does the equation $\sqrt{x+5} + x = 10$ have?

   (a) 0.
   (b) 1.
   (c) 2.
   (d) 3.
   (e) 4.

18. A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. They fought over who should get the extra coins, and one pirate was killed. When the remaining pirates divided the coins into equal pieces, 10 coins were left over. Another brawl ensued, and yet another pirate was slain. But now the fortune was evenly distributed among the survivors. What was the least number of coins that could have been stolen?

    (a) 2570.
    (b) 2910.
    (c) 3450.
    (d) 3930.
    (e) 4360.

19. An 8×8 checkerboard has its corners trimmed to form a circular dartboard with diameter equal to the original length of the side of the board. A dart is thrown at the circle randomly. What is the probability that the dart lands in a square that is entirely contained in the circle?

    (a) 32%.
    (b) 50%.
    (c) 64%.
    (d) 75%.
    (e) 76%.

20. If x is a real number such that $x^2 > 6x - 9$, then how many numbers can x not be equal to?

    (a) 0.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) 4.

21. Let $ABCD$ be a quadrilateral. The diagonals $\overline{AC}$ and $\overline{BD}$ meet at O. Suppose S_1, S_2, S_3, and S_4 are the areas of triangles $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle AOD$, respectively. If $S_1 = 2S_2$ and $S_2 + 2 = S_3 + 5 = S_4$, the area of $ABCD$ is

    (a) 8.
    (b) 16.
    (c) 39.
    (d) 45.
    (e) not enough information.

22. Given that $\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1$. What is $\sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$?

    (a) $np(1-p)$.
    (b) np.
    (c) n.
    (d) $\dfrac{n(n+1)}{2}$.
    (e) $\dfrac{n^2(n+1)}{2}$.

23. Let $F = .56363...$ be an infinite repeating decimal with the digits 6 and 3 repeating. When F is written as a fraction in lowest terms, the denominator exceeds the numerator by

(a) 24.
(b) 31.
(c) 48.
(d) 436.
(e) 558.

24. On a clock, what angle does the minute hand make with the hour hand at 12:28? Assume that the hour hand stays in the "12" position.

(a) 28°.
(b) 84°.
(c) 168°.
(d) 280°.
(e) 336°.

25. Let $x_1 = 2$ and

$$x_{n+1} = x_n + \frac{1}{x_n}$$

for $n \geq 1$. Which interval contains x_{1000}?

(a) $(0, 1)$.
(b) $(1, 2)$.
(c) $(2, 1000)$.
(d) $(1000, 1002)$.
(e) $(1002, 2000)$.

26. The roots of $Ax^2 + Bx + C = 0$ are r and s. For the roots of

$$x^2 + px + q = 0$$

to be r^2 and s^2, p must equal:

(a) $\dfrac{B^2 - 4AC}{A^2}$.
(b) $\dfrac{B^2 - 2AC}{A^2}$.
(c) $\dfrac{2AC - B^2}{A^2}$.
(d) $B^2 - 2C$.
(e) $2C - B^2$.

27. One hundred bushels of grain are distributed among 100 people in such a way that each man receives 3 bushels, each woman 2 bushels, and each child $\frac{1}{2}$ bushel. One possibility is that there are 2 men, 30 women, and 68 children. How many other possibilities are there if there is at least one man, one woman, and one child?

(a) 1.
(b) 2.
(c) 3.
(d) 4.
(e) 5.

28. Suppose that a, b, c, are positive numbers, and that $x > 0$ and $y > 0$. The expression

$$\log\frac{a}{b} + \log\frac{b}{c} + \log\frac{c}{d} - \log\frac{ay}{dx}$$

can be reduced to:

(a) 0.
(b) 1.
(c) $\log\frac{y}{x}$.
(d) $\log\frac{x}{y}$.
(e) $\log\frac{a^2y}{d^2x}$.

29. A standard deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Define the events E_i, $i = 1, 2, 3, 4$, by $E_1 = \{$the first pile has exactly 1 ace$\}$, $E_2 = \{$the second pile has exactly 1 ace$\}$, $E_3 = \{$the third pile has exactly 1 ace$\}$, $E_4 = \{$the fourth pile has exactly 1 ace$\}$.

What is the probability that E_1 *and* E_2 *and* E_3 *and* E_4 occur?

(a) $\dfrac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \cdot \dfrac{\binom{3}{1}\binom{28}{12}}{\binom{31}{13}} \cdot \dfrac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \cdot \dfrac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}}$.

(b) $\dfrac{\binom{4}{4}\binom{48}{48}}{\binom{52}{4}}$.

(c) $\dfrac{1}{4}$.

(d) $\dfrac{\binom{4}{1}\binom{51}{12}}{\binom{52}{13}} \cdot \dfrac{\binom{3}{1}\binom{38}{12}}{\binom{39}{13}} \cdot \dfrac{\binom{2}{1}\binom{26}{12}}{\binom{26}{13}} \cdot \dfrac{\binom{1}{1}\binom{13}{13}}{\binom{13}{13}}$.

(e) $\dfrac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \cdot \dfrac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \cdot \dfrac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \cdot \dfrac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}}$.

30. How many roots does $\sin(1/x)$ have where $0.001 \leq x \leq 0.01$?

(a) 287.
(b) 301.
(c) 499.
(d) 599.
(e) 899.

31. A drawer has 10 red socks and 10 blue socks. If 3 socks are selected from the drawer, without replacement, what is the probability that they will all be the same color?

(a) 0.0225.
(b) 0.1500.
(c) 0.1667.
(d) 0.2105.
(e) 0.3000.

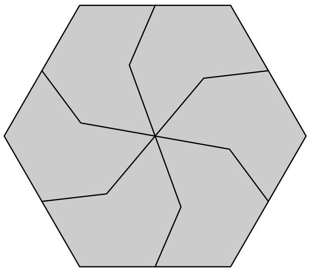
32. If the base of a rectangle is increased by 10% and the area is unchanged, then the height is decreased by:

    (a) 9%.
    (b) $9\frac{1}{11}$%.
    (c) 10%.
    (d) 11%.
    (e) $11\frac{1}{9}$%.

33. How many integers n satisfy the equation $1/a + 1/b = n/(a + b)$ for some non-zero integer values of a and b (with $a + b \neq 0$)?

    (a) 0.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) more than 3.

34. Determine the smallest angle of rotational symmetry of the given figure.



    (a) 45 degrees.
    (b) 60 degrees.
    (c) 90 degrees.
    (d) 180 degrees.
    (e) no rotation symmetry.

35. The limit of the sum of an infinite number of terms in a geometric progression is $\frac{a}{1-r}$ where a denotes the first term and $-1 < r < 1$ denotes the common ratio. The limit of the sum of their squares is:

    (a) $\frac{a^2}{(1-r)^2}$.
    (b) $\frac{a^2}{1+r^2}$.
    (c) $\frac{a^2}{1-r^2}$.
    (d) $\frac{4a^2}{1+r^2}$.
    (e) none of these.