# Comprehensive Mathematics Contest

Elon University Mathematics and Statistics Department

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## Multiple Choice

1. In degrees, what is the measure of an interior angle of a regular icosagon (20 sides)?
   (a) 144.
   (b) 153.
   (c) 162.
   (d) 171.
   (e) none of the above.

2. What is the last digit in the decimal representation of $2023^{2023}$?
   (a) 1.
   (b) 3.
   (c) 5.
   (d) 7.
   (e) 9.

3. Suppose $3f(x) - f(2-x) = x^2$. What is $f(x)$?
   (a) $x^2 + 4x + 4$.
   (b) $\dfrac{x^2 + 4x + 4}{8}$.
   (c) $\dfrac{x^2 - x + 1}{2}$.
   (d) $x^2 - 4x + 4$.
   (e) none of the above.

4. If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of $p$ and $q$, $\log_{10} 3$ equals
   (a) $pq$.
   (b) $\dfrac{3p}{3pq+1}$.
   (c) $\dfrac{3q}{2pq+1}$.
   (d) $\dfrac{3pq}{2pq+1}$.
   (e) $p+q$.

5. When 20! is written in base 12, how many 0's are at the end?
   (a) 4.
   (b) 5.
   (c) 6.
   (d) 7.
   (e) 8.

6. How many integers between 2023 and $2023^2$ are perfect squares and have sum of digits equal to 14?
   (a) 0.
   (b) 1.
   (c) 3.
   (d) 4.
   (e) 6.

7. What is the remainder when $4^{2020} + 6^{2020}$ is divided by 25?
   (a) 1.
   (b) 2.
   (c) 3.
   (d) 4.
   (e) 5.
8. Lauren runs around a track; she completes 1 lap in 90 seconds. Rachel runs in the opposite direction. They meet every 40 seconds. How many seconds does it take Rachel to complete 1 lap?
   (a) 60.
   (b) 68.
   (c) 72.
   (d) 82.
   (e) none of the above.
9. There are two non-congruent rectangles with integer side lengths whose area is equal to its perimeter. What is the absolute value of the difference of their areas?
   (a) 2.
   (b) 4.
   (c) 6.
   (d) 8.
   (e) none of the above.
10. What is the area of a triangle with sides 10, 13, and 13?
    (a) 50.
    (b) 60.
    (c) 65.
    (d) 72.
    (e) 130.
11. What is the probability of randomly picking a value $b$ from the interval [-20,15] so that the equation $2x^2 + bx + 8 = 0$ has at least one real solution?
    (a) 0.
    (b) 16/35.
    (c) 19/35.
    (d) 24/35.
    (e) none of the above.
12. How many triangles can be constructed with seven equal length sticks in such a way that the perimeter of each triangle is the total length of the seven sticks?
    (a) 0.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) 4.

13. The equation $\dfrac{2}{x+3} + \dfrac{1}{x-2} = \dfrac{-10}{x^2+x-6}$ has
    (a) no solutions.
    (b) one solution.
    (c) two solutions.
    (d) three solutions.
    (e) More than 3 solutions.

14. How many values of $m$ produce exactly one root to the equation $(x-m)^2 + 3x = 0$.
    (a) 0.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) More than 3.

15. A number $x$ is picked at random from the interval [0,1]. What is the probability that $|4x-1| + |4x-2| = 1$?
    (a) 0.
    (b) 1/4.
    (c) 1/2.
    (d) 2/3.
    (e) 3/4.

16. How many polynomials $p(x)$ satisfies $xp(x-2) = (x-11)p(x)$ and $p(12) = 12^2$?
    (a) 0.
    (b) 1.
    (c) 2.
    (d) infinitely many.
    (e) none of the above.

17. Suppose $sin(x) + cos(x) = 1.4$. what is the value of $sin(2x)$?
    (a) 0.4.
    (b) 0.86.
    (c) 0.96.
    (d) 0.98.
    (e) 1.

18. Suppose $S$ is the sum and $P$ is the product of all the roots for the polynomial $3x^3 + 6x^2 - 12x + 42$. Determine $S - P$
    (a) 10.
    (b) 12.
    (c) 14.
    (d) 22.
    (e) 36.

19. Suppose $a + b = a^2 + b^2 = \dfrac{\sqrt{2}}{2}i$. Compute absolute value of the real part of $a$.
    (a) $\frac{1}{2}$.
    (b) $\frac{\sqrt{2}}{2}$.
    (c) 1.
    (d) $\sqrt{2}$.
    (e) 2.
20. How many positive integers less than 2020 have no repeating digits?
    (a) 1105.
    (b) 1242.
    (c) 1245.
    (d) 1249.
    (e) none of the above.
21. Let $p(x)$ be a $6^{th}$ degree polynomial that has a minimum value of -7 which occurs at $x = 1$, $x = 3$ and $x = 5$. If $p(2) = 2$ compute $p(4)$.
    (a) 2.
    (b) 144.
    (c) 204.
    (d) 218.
    (e) 256.
22. Let $a$ and $b$ be positive integers with $\frac{2}{3} < \frac{a}{b} < \frac{3}{4}$. What is the smallest value $|a + b|$ can obtain?
    (a) 8.
    (b) 9.
    (c) 12.
    (d) 14.
    (e) 18.
23. If the ratio of $x - y$ to $x + y$ is 3/4 what is the ratio of $y$ to $x$?
    (a) 1.
    (b) 7.
    (c) 1/7.
    (d) 3/4.
    (e) 4/3.
24. Let $f(x)$ be the minimum of the numbers $3x + 1$, $x + 1$, and $5 - 2x$. What is the maximum of $f(x)$?
    (a) 4/3.
    (b) 4/5.
    (c) 17/5.
    (d) 7/3.
    (e) 1.
25. The sum of an infinite geometric series with common ration $|r| < 1$ is 3, and the sum of the squares of the terms is 4.5. What is the first term of the series?
    (a) 1.
    (b) 1.5.
    (c) 2.
    (d) 3.
    (e) 3.5.

## Integer Answers

26. For what real number $c$ does the equation $|x^2 - 8x + 15| = c$ have exactly 3 real solutions?

27. How many ordered pairs of integers $(x, y)$ satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$?

28. A function $f$ is defined recursively by
$$f(k+1) = \frac{af(k)}{2a + f(k)},$$
where $a$ is a real number, $f(1) = 4$ and $f(5) = 1$. Find $|a|$.

29. Let $a$ and $b$ be integers and $2a + 2b = 4 + 2ab = a^2 - b^2$. Find $a$.

30. One generalization of the Fibonacci numbers, called the Tribonacci numbers, $T_n$, are defined as follows: $T_0 = 0, T_1 = 1$, and $T_2 = 1$. For all $n \geq 3$, we have $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. Compute the smallest Tribonacci number greater than 100 which is prime.