THE MATHEMATICAL STRUCTURE OF THE LAW

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Scientific "law" and human-made law ("social law") are both "laws" in a general sense—scientific laws "govern" the workings of the material world and social laws govern the behavior of people. Beyond this superficial resemblance, do social laws partake of the same sorts of mathematical structures as scientific laws? Many theorists have proposed formal, deontic-oriented logical models of legal rights and other entitlements. Here, leveraging the typology of Wesley Hohfeld, this Article proposes a novel, mathematical model of legal entitlements. This model incorporates physical and mathematical properties—such as entropy, indeterminacy, temperature, and modularity—to measure quantitative properties of legal systems. Moreover, this Article proposes a post-classical approach to model ontological legal indeterminacy by adapting the formalism of quantum mechanics. These understandings have important implications for the nature of legal rules, legal AI, game theory and the law, and the ontology of rule-based systems. Of particular note, the formalism suggests a novel approach to the quantum measurement problem, proposing that measurement is a "second-order" physical process, which is fundamentally different from the "first-order" physical processes currently described by quantum mechanics.

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INTRODUCTION

Scientific law and human-made law ("social law") apply to quite different domains and are expressed in quite different terms.[1] Scientific laws describe the structure and dynamics of the material world, typically in mathematical terms.[2] Social laws govern the behavior of people, typically in non-mathematical terms.[3] Yet, in at least a metaphorical sense, one might characterize how social laws govern people as roughly akin to how scientific laws "govern" matter and fields.[4]

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[1] *See generally* KARL POPPER, THE OPEN SOCIETY AND ITS ENEMIES 56–57 (one-vol. ed. 2020) (1945) (contrasting "normative" and "natural" laws); H.L.A. HART, THE CONCEPT OF LAW 187 (3d ed. 2012) (commenting on J.S. Mill's view of the "perennial confusion between laws which formulate the course or regularities of nature, and laws which require men to behave in certain ways"); Andreas Rahmatian, *The Nature of Laws in Law and in Economics*, *in* LOIS DES DIEUX, DES HOMMES ET DE LA NATURE: ÉLÉMENTS POUR UNE ANALYSE TRANSVERSALE 109, 109–42 (Giuseppe Longo ed., 2017) ("[L]egal rules are not laws in the sense of natural laws and as natural scientists would understand them."); Stephen A. Siegel, *John Chipman Gray and the Moral Basis of Classical Legal Thought*, 86 IOWA L. REV. 1513, 1519–23 (2001) (discussing how classical legal scholars perceived "differences" between "legal science" and "natural science").

[2] *See Excerpts from The Assayer*, *in* DISCOVERIES AND OPINIONS OF GALILEO 229, 237–38 (Stillman Drake trans., 1957) (1623) (contending that the "book of nature . . . is written in the language of mathematics"); PETER MITTELSTAEDT & PAUL A. WEINGARTNER, LAWS OF NATURE 5 (2005) ("Every law of nature is a representation of some structure of nature (i.e. of the real world)."). *See generally* D.M. ARMSTRONG, WHAT IS A LAW OF NATURE? 6 (1983) ("[T]he scientific theories which we now work with are obviously a reasonable approximation to at least some of the real laws of nature."); ALEXANDER BIRD, NATURE'S METAPHYSICS: LAWS AND PROPERTIES 1–5 (2007) (describing a variety of philosophical viewpoints regarding the metaphysical nature of natural laws).

[3] HANS KELSEN, GENERAL THEORY OF LAW AND STATE 3 (Anders Wedberg trans., 1945) ("Law is an order of human behavior. An 'order' is a system of rules."); HART, *supra* note 1, at 67 (conceiving of law as a social rule that "constitutes a standard of behaviour for the group"); Roscoe Pound, *My Philosophy of Law*, *in* MY PHILOSOPHY OF LAW: CREDOS OF SIXTEEN AMERICAN SCHOLARS 249–62 (1941) (arguing that "law" is "a highly specialized form of social control in a developed politically organized society—a social control through the systematic and orderly application of the force of such a society").

[4] *See* Christopher Langdell, Professor, Harvard Law School, Address at the Celebration of the Two Hundred and Fiftieth Anniversary of the Founding of Harvard College (Nov. 5, 1886), *in* REPORT OF THE ORGANIZATION AND OF THE FIRST GENERAL MEETING 49–50 (1887), *reprinted in* 3 LAW Q. REV. 123, 124 (1887) (contending that "law is a science, and that all the available materials of that science are contained in printed books"); Harold J. Berman & Charles J. Reid, Jr., *The Transformation of English Legal Science: From Hale to Blackstone*, 45 EMORY L.J. 437, 498–504 (1996) (explaining the "[s]imilarities between the empirical method of the natural sciences and the empirical method of the new legal science" of England in the 17th and 18th centuries); Howard Schweber, *The "Science" of Legal Science: The Model of the Natural Sciences in Nineteenth-Century American Legal Education*, 17 L. & HIST. REV. 421, 455–64 (1999) (exploring the relationship of "natural science" and "legal science" in the 1870s). *See generally* Rahmatian, *supra* note 1, at 109 ("With the evolution of the modern natural sciences in the seventeenth and eighteenth centuries the natural laws which the sciences discovered also became a model for the understanding of 'social laws' and the development of social and legal institutions."); 1 WILLIAM BLACKSTONE, COMMENTARIES ON THE LAWS OF ENGLAND 38–39

Beyond this superficial resemblance, is it possible to formalize social laws using the same sorts of mathematical structures as scientific laws? Here, I propose a novel mathematical model of legal entitlements,[5] contending that social laws adhere to a similar mathematical formalism as scientific laws in a "structural" sense.[6] By "structural," I refer to the underlying mathematical structure that describes the relation of laws to the objects they "govern."[7] For instance, the laws of Newtonian physics

(Chicago, Callaghan & Co. 2d rev. ed. 1876) (1783) ("Law, in its most general and comprehensive sense, signifies a rule of action; and is applied indiscriminately to all kinds of action, whether animate or inanimate, rational or irrational. Thus we say, the laws of motion, of gravitation, of optics, or mechanics, as well as the laws of nature and of nations. And it is that rule of action, which is prescribed by some superior, and which the inferior is bound to obey.").

[5] *See infra* Part III (setting forth a mathematical formalism of Hohfeldian legal relations). Although prior scholars have argued that social law should resemble scientific law, none have introduced a mathematical formalism describing the notion of "law" in both disciplines. *Cf.* Rahmatian, *supra* note 1, at 109 ("Juristic laws . . . should have had qualities emulating or approximating the laws of physics."). In contrast, legal scholars have regularly applied generalized mathematical techniques used in the sciences, but this aspect of legal analysis differs from a generalized, mathematical treatment of legal relations themselves. *See, e.g.*, James Ming Chen, *Legal Quanta: A Mathematical Romance of Many Dimensions*, 2016 MICH. STATE L. REV. 313, 313 (2016) (noting how a recent symposium "demonstrates several distinct applications of mathematics to law and the use of quantitative techniques to model, describe, and predict legal phenomena"); Harry Surden, *Artificial Intelligence and Law: An Overview*, 35 GA. ST. U. L. REV. 1305, 1326–28 (2019) (describing a variety of "computer and mathematical techniques" used in the emerging field of AI & Law).

[6] *See infra* Parts V–VI (contrasting the "structure" and "content" of the law and positing that the mathematical structure of social law and scientific law is one and the same). Notably, in economics, an entire branch of "econophysics" has not only borrowed mathematical techniques from physics, but has posited a deeper connection between the nature of economic and physical law. *See* ROSARIO N. MANTEGNA & H. EUGENE STANLEY, AN INTRODUCTION TO ECONOPHYSICS: CORRELATIONS AND COMPLEXITY IN FINANCE (2000) (examining the application of methods in physics to model financial phenomena); Anirban Chakraborti et al., *Econophysics Review: II. Agent-Based Models*, 11 QUANTITATIVE FIN. 1013 (reviewing applications of physics to agent-based models of economics). And the growing field of "quantum social science" examines how quantum mechanics informs the nature of political science, sociology, psychology, and related disciplines. *See* ALEXANDER WENDT, QUANTUM MIND AND SOCIAL SCIENCE (2015) (contesting social science's implicit assumption that reality is classical in nature).

[7] *See infra* Part V (distinguishing "structural" from "synthetic" aspects of the law). Importantly, this article concerns the structural, logical formalism that any law must adhere to, rather than the substance of the laws themselves. *See id.* Like Hohfeld, the treatment here is agnostic as to how the substance of the law comes about. *See id.* Thus, the thesis presented here does not sound in "natural law," which in its most robust form posits that "legal and social rules [are] 'discovered,' not made, and the rules in their ideal form [are] supposed to be mathematically calculable and predictable cause-effect relations." Rahmatian, *supra* note 1, at 109.

Nor is the approach here "formalist" in the substantive sense, because it does not argue that judges, lawyers, or others in the legal system should necessarily use logic, mathematics, or the like to determine the ultimate substance of the law. *Cf.* CESARE BECCARIA, DEI DELITTI E DELLE PENE [ON CRIMES & PUNISHMENTS] 14–15 (Seven Treasures Publ'ns 2009) (1764) (contending in an early "formalist" work that "[i]n every criminal case the judge should reason syllogistically" concern the motion of idealized particles,[8] and criminal laws concern the permissible actions of legal persons.[9] The mathematical framework introduced here applies equally to describe the structural relation between a scientific object (e.g., an idealized particle) and the object's governing laws (e.g., Newton's laws) as well as a legal object (e.g., a legal person) and its governing laws (e.g., criminal laws).[10]

Many scholars have proposed logical, but not precisely mathematical,[11] formalisms of the underlying structure of legal entitlements.[12] Arguably, the earliest legal theorist to coherently and fairly

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in a formal sense to determine whether the accused is guilty or not). *See generally* M. H. Hoeflich, *Law & Geometry: Legal Science from Leibniz to Langdell*, 30 AM. J. LEGAL HIST. 95 (1986) (exploring the historical lineage of formalist view of legal reasoning as embodied in "legal science").

Although this article's approach may be categorized as "formalist" from a structural perspective, *see infra* Part V, notably, the legal realists—who eschewed the natural law and formalist traditions—were the primary early adopters of the Hohfeldian approach. *See* Pierre Schlag, *How to Do Things with Hohfeld*, 78 L. & CONTEMP. PROBS. 185, 189 (2015) (noting that "Arthur Corbin [was] one of Hohfeld's early legal realist enthusiasts"). Relatedly, common arguments against the use of mathematical principles in the law would not apply here since those arguments are invariably directed towards the use of mathematical techniques to derive the content or inform the substance of the law or to make evidentiary inferences. *See* Laurence H. Tribe, *Trial by Mathematics: Precision and Ritual in the Legal Process*, 84 HARV. L. REV. 1329, 1353–55 (1971) (critiquing the use of Bayes' Theorem in evidentiary determinations); HENRY M. HART, JR. & ALBERT M. SACKS, THE LEGAL PROCESS 107–10 (1994) (dismissing the notion that law should be viewed through the lens of natural science).

[8] *See* STEPHEN T. THORNTON & JERRY B. MARION, CLASSICAL DYNAMICS OF PARTICLES AND SYSTEMS 57–61 (2013) (describing Newton's laws of motion).

[9] *See* Paul H. Robinson, *A Functional Analysis of Criminal Law*, 88 NW. U. L. REV. 857, 857 (1994) (noting that criminal law provides "rules of conduct" that "provide ex ante direction to members of the community as to the conduct that must be avoided (or that must be performed) upon pain of criminal sanction"); Joshua Kleinfeld, *Reconstructivism: The Place of Criminal Law in Ethical Life*, 129 HARV. L. REV. 1485, 1490 (2016) (arguing that criminal law is "an instrument of normative reconstruction" that operates through "condemnatory punishment").

[10] *See infra* Part VI (describing how the mathematical formalism of Hohfeldian legal relations also applies to physical relations).

[11] Logic is generally considered a branch of mathematics. *See* Stewart Shapiro, *Logical Consequence, Proof Theory, and Model Theory*, *in* THE OXFORD HANDBOOK OF PHILOSOPHY OF MATHEMATICS AND LOGIC 651 (Stewart Shapiro ed., 2005) ("Since at least the beginning of the twentieth century . . . logic has become a branch of mathematics . . . ."). However, here I refer to "mathematical" in the narrower and more common sense of a mathematical formalism that admits of quantitative calculation. *See* Lisa Shabel, *Apriority and Application: Philosophy of Mathematics in the Modern Period*, in THE OXFORD HANDBOOK OF PHILOSOPHY OF MATHEMATICS AND LOGIC 28 (Stewart Shapiro ed., 2005) (noting that in the modern period, "[m]athematics was understood to be the science that systematized our knowledge of magnitude, or quantity").

[12] *See, e.g.*, Layman E. Allen, *Formalizing Hohfeldian Analysis to Clarify the Multiple Senses of 'Legal Right': A Powerful Lens for the Electronic Age*, 48 S. CAL. L. REV. 428 (1974) (introducing a logical formalism for the Hohfeldian relations and related propositions); Layman E. Allen & Charles S. Saxon, *Achieving Fluency in Modernized and Formalized Hohfeld: Puzzles and Games for the Legal Relations Language*, ICAIL '97 PROC. OF THE 6TH INT'L CONF.

accurately describe the formal structure of the law was the early 20th century legal philosopher, Wesley Hohfeld.[13] Specifically, Hohfeld posited that there are eight logically related "fundamental legal relations" that can be combined to describe all legal phenomena.[14] Later scholars used Hohfeld's typology to develop even more formal, deontic logic-oriented theories of legal systems.[15] These theories generally encompass three key elements: legal actors, legal relations, and knowledge about states of the world concerning the legal relations and actors.[16] Legal actors typically comprise natural and non-natural persons subject to the laws of a given legal system.[17] Legal relations define whether the legal actors may engage (or not) in particular conduct under the applicable legal rules, and whether the actors can alter (or not) those legal rules.[18] Knowledge about states of

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ON A.I. AND L. 19 (describing the "A-HOHFELD" language for formalizing the Hohfeldian relations); Stig Kanger, *Law and Logic*, 38 THEORIA 105 (1972) (offering a logical representation of the legal relations grounded on modal and deontic logic); LARS LINDAHL, POSITION AND CHANGE: A STUDY IN LAW AND LOGIC (D. Reidel Publ'g Co. 1977) (examining a variety of logical formalisms of Hohfeld's and others' conceptions of legal relations) .

[13] *See* David Frydrych, *Rights Correlativity*, *in* WESLEY HOHFELD A CENTURY LATER: EDITED WORK, SELECT PERSONAL PAPERS, AND ORIGINAL COMMENTARIES (Shyam Balganesh, Ted M. Sichelman & Henry E. Smith eds., 2022) (exploring the historical lineage of the Hohfeldian relations); Joseph William Singer, *The Legal Rights Debate in Analytical Jurisprudence from Bentham to Hohfeld*, 1982 WIS. L. REV. 975, 1049–50 & n.210 (1983) (examining the similarities and differences between Hohfeld's typology and the earlier one of Salmond).

[14] Wesley Newcomb Hohfeld, *Some Fundamental Legal Conceptions as Applied in Judicial Reasoning*, 23 YALE L.J. 16 (1913) (positing a system of first- and second-order jural relations) [hereinafter Hohfeld, *Fundamental Legal Conceptions*]; *see also* Ted M. Sichelman, Annotation, *Wesley Hohfeld's Some Fundamental Legal Conceptions as Applied in Judicial Reasoning*, *in* WESLEY HOHFELD A CENTURY LATER: EDITED WORK, SELECT PERSONAL PAPERS, AND ORIGINAL COMMENTARIES (Shyam Balganesh, Ted M. Sichelman & Henry E. Smith eds., 2022) [hereinafter Sichelman, *Annotated Fundamental Legal Conceptions*] (offering detailed annotations further explaining the Hohfeldian typology and related issues in contemporary jurisprudence).

[15] *See supra* note 12 (citing references).

[16] *See, e.g.*, Allen, *supra* note 12, at 433–34 ("Hohfeld, in effect, specified the term 'right' to refer to a three-term relationship between two persons and an action-the right-holder, the other party, and an act of the other party."); LINDAHL, *supra* note 12, at 29.

[17] Hohfeld believed that legal actors could only be individuals. *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 44. However, modern approaches generally view the corporation and other entities as actors effectively subject to legal relations. *See* James S. Coleman, *Responsibility in Corporate Action: A Sociologist's View*, *in* CORPORATE GOVERNANCE AND DIRECTORS' LIABILITIES LEGAL, ECONOMIC AND SOCIOLOGICAL ANALYSES ON CORPORATE SOCIAL RESPONSIBILITY 69, 72 (Klaus J. Hopt & Gunther Teubner eds., 1985) ("[T]he legal system constrains the actions not only of natural persons but also corporate actors.").

[18] Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 32–33 (exploring the nature of legal obligations and privileges); *id.* at 44–45 (discussing the "power" to alter legal rules).

the world includes any relevant, factual proposition related to the actors and relations.[19]

These logical formalisms have proved very useful but suffer from at least two important limitations. First, they do not easily connect with the rich body of mathematics, physics, information theory, and other scientific disciplines that model the properties of complex systems.[20] For instance, deontic formalisms do not easily adapt to the well-defined concepts of entropy, temperature, information content, and the like, so as to describe the properties of legal systems in a precise quantitative fashion.[21]

Second, nearly all of these logical renderings of legal relations implicitly make two key assumptions: (1) all of the information relevant to determining whether a given legal actor is subject to a specific legal relation is knowable;[22] and (2) the boundaries of legal relations are well-defined such that an external observer who knows all the relevant information can determine whether or not the given legal actor is subject to the relation.[23] In other words, these approaches assume that a fully knowledgeable

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[19] *See* J. Wolenski, *Deontic Logic and Possible Worlds Semantics: A Historical Sketch*, 49 STUDIA LOGICA 273, 277–78 (1990) (describing the relationship between deontic logic and Kripke's conception of "possible worlds" and "real worlds").

[20] Although deontic logic is closely related to modal logic, and modal logic has played an important role in the philosophy of physics, there has been little mathematization of modal logic so as to connect it more closely to the mathematical foundation of physics and information theory. *See, e.g.*, M.L. Dalla Chiara, *Quantum Logic and Physical Modalities*, 6 J. PHIL. LOGIC 391 (1977) (examining the relationship between modal and quantum logic); David Deutsch & Chiara Marletto, *The Constructor Theory of Information*, CTR. FOR QUANTUM COMPUTATION, UNIV. OF OXFORD (2014) (proposing "a theory of information expressed solely in terms of which transformations of physical systems are possible and which are impossible"). The major notable exception is August Stern, who proposed a mathematical formulation of logic, including modal logic, and related it to principles in physics, though his approach has been infrequently recognized. *See, e.g.*, AUGUST STERN, MATRIX LOGIC (1988). Even still, there has been no mathematization of deontic logic and the related Hohfeldian typology, so as to provide a mathematical formulation of social law. *See supra* note 56 and accompanying text.

[21] *See* Ted M. Sichelman, *Quantifying Legal Entropy*, 9 FRONTIERS IN PHYSICS 264 (2021) [hereinafter Sichelman, *Legal Entropy*] (relying partly on the concepts introduced herein to formulate a model of the "entropy" of legal relations in "Hohfeldian space"); *see infra* Part IV (explaining how legal entropy and temperature helps to explain the modularization of the law and its limits).

[22] *See* Guido Governatori et al., *A Defeasible Logic for Modelling Policy-Based Intentions and Motivational Attitudes*, 17 LOGIC J. IGPL 227, 231–34 (2009) (noting the assumption of "factual omniscience" in deontic logic). *See generally* Jennifer Rose Carr, Should You Believe the Truth? (2020) (Working Paper), https://pages.ucsd.edu/~j2carr/pdfs/truthnorm.pdf ("The TRUTH NORM [i.e., that 'ought to believe all and only truths], on the intended interpretation, doesn't say that you should *become* omniscient. It says that you should already *be* omniscient.").

[23] *See* Governatori et al., *supra* note 22, at 231 (noting that for an agent with "intentions and obligations . . . for every proposition A she knows whether A or A").

observer can do better than a merely probabilistic assessment of legal rights and other entitlements—rather, legal relations exhibit.[24]

For example, suppose that legal actor *O* is the owner of a piece of land *L*.[25] Under the common law of property, a third-party *T* with no ownership interest in *L*, and that has no sufficient legal excuse to enter *L*, will be subject to a Hohfeldian *duty* (owed to *O*) not to enter *L*.[26] Under classical formalism, *T* cannot be subject to such a duty not to enter *L* and simultaneously have *no* duty not to enter *L*.[27] Stated another way, one cannot be subject a duty not to trespass if one has *no* duty not to trespass. Indeed, holding such conflicting rights would, on the standard deontic view, violate fundamental logic.[28] Classical theorists do admit that, in practice, whether or not a given legal actor is subject to a specific legal

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[24] Bertram Malle, From Binary Deontics to Deontic Continua: The Nature of Human (and Robot) Norm Systems (Feb. 2018) (Working Paper), https://research.clps.brown.edu/SocCogSci/Publications/Pubs/Malle_2018_Deontic_Continua_Vienna.pdf ("[V]irtually all formal representations of normative systems . . . have suffered from one major limitation: binary deontic concepts.") Even most formulations that admit of concerns with "factual omniscience" of agents in a deontic system still adhere to the binary approach to deontic modalities. *See* Governatori et al., *supra* note 22, at 248–56 (adopting a defeasible logic that incorporates binary deontic modalities to address concerns of omniscience). And the few approaches that reject "binary deontic concepts" implement a purely "continuous" or "fuzzy" set of norms, rather than one that allows transitions from a fuzzy norm to a binary norm upon judgment. *See* Matthias Nickles, *Towards a Logic of Graded Normativity and Norm Adherence*, *in* NORMATIVE MULTI-AGENT SYSTEMS, (G. Boella et al. eds., 2007) (proposing a probabilistic belief logic to norms grounded in the expectation of enforcement); *see also* Oren Perez, *Fuzzy Law: A Theory of Quasi-Legal Systems*, 28 CAN. J. L. & JURIS., 343 (2015) (proposing a normative continuum to explain "soft laws" that emanate from non-state actors).

[25] *See* A.M. Honoré, *Ownership*, *in* OXFORD ESSAYS IN JURISPRUDENCE 107, 107–28 (A.G. Guest ed., 1961) (listing the elements of the bundle of rights that constitutes property ownership).

[26] *See* J.M. Balkin, *The Hohfeldian Approach to Law and Semiotics*, 44 U. MIAMI L. REV. 1119, 1142 (1990) ("For example, my claim right (a Hohfeldian right) to the exclusive possession of my property carries with it a duty on the part of others not to trespass.").

[27] *See* LAMBER M.M. ROYAKKERS, EXTENDING DEONTIC LOGIC FOR THE FORMALISATION OF LEGAL RULES 70–71 (Fransisco J. Laporta, Aleksander Peczenik & Frederick Schauer eds., 1998) (analyzing the standard deontic logic assumption that there are no personal conflicting obligations). For a deontic logic that admits of conflicting duties, *see* E.J. Lemmon, *Deontic Logic and the Logic of Imperatives*, 8 LOGIQUE ET ANALYSE 39, 43–51 (1965).

[28] From a classical logic perspective, a legal proposition A must be either true or false, and if A is true, then not-A is false, and vice-versa. *See* Stephen Munzer, *Validity and Legal Conflicts*, 82 YALE L.J. 1140, 1162–66 (1973) (examining the standard view "that conflicting norms are logically inconsistent"); IMMANUEL KANT, GROUNDWORK OF THE METAPHYSICS OF MORALS 16 (Mary Gregor ed. & trans., 1998) (1797) (asserting that conflicting duties is "inconceivable"). *See generally* Matthew H. Kramer, *Rights Without Trimmings*, *in* A DEBATE OVER RIGHTS: PHILOSOPHICAL INQUIRIES 7, 10–20 (Matthew H. Kramer, N.E. Simmonds & Hillel Steiner, eds., 1998) (examining the conflicting duties in view of the Hohfeldian formalism and noting that the formalism does not per se preclude such conflicts).

relation may be epistemologically indeterminate.[29] Yet, such indeterminacy results merely from a lack of complete information needed to assess whether the actor is subject to the relation.[30]

Post-classical legal theorists, especially those associated with the critical legal studies movement, impugned the classical approach's determinism.[31] In particular, these theorists argued that indeterminacy is an inherent feature of law, such that—in a large share of legal situations—it is impossible *in principle* to determine with certainty whether a given legal actor is subject to a given legal relation or not.[32] More classically minded theorists argued that this inherent indeterminacy was confined to so-called "hard cases,"[33] but whatever the extent of the indeterminacy, scholars generally perceive it as undermining notions of law's "rationality" held under the classical approach.[34]

This Article makes three novel contributions to the literature. First, I offer a mathematical formulation of the Hohfeldian relations that easily lends itself to the rich body of quantitative measures in mathematics, physics, information theory, and other scientific disciplines in order to describe the properties of legal systems more accurately and precisely.[35]

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[29] *See* Ken Kress, *A Preface to Epistemological Indeterminacy*, 85 NW. U. L. REV. 134, 138–39 (1990) ("Metaphysical indeterminacy speaks to whether there is law; epistemic indeterminacy, to whether the law can be known. . . . [M]etaphysical determinacy is compatible with great epistemic indeterminacy."). In this regard, Kress further posits that "a natural law theorist is likely to believe in substantial metaphysical determinacy in law [but] may well believe in substantial or radical epistemic indeterminacy." *Id.*

[30] *See id.* at 139 (cataloguing various senses of epistemic indeterminacy including "the right answer is determinable in principle").

[31] *See, e.g.*, Max Radin, *A Restatement of Hohfeld*, 51 HARV. L. REV. 1141, 1146–63 (1938) (interpreting Hofeldian analysis through the lens of legal realism); Duncan Kennedy, *Legal Formality*, 2 J. LEGAL STUD. 351, 364 (1973) (contesting that the law is "inherently certain and predictable"); Roberto Unger, *The Critical Legal Studies Movement*, 96 HARV. L. REV. 561, 567–76 (1983) (critiquing objectivism and formalism in the law).

[32] *See* Kress, *supra* note 29, at 138–39 (noting that his arguments in prior work, which addresses the radical indeterminacy thesis of critical legal realists and others, addressed "metaphysical indeterminacy"); *see also* Ken Kress, *Legal Indeterminacy*, 77 CALIF. L. REV. 283, 286 (1989) [hereinafter Kress, *Legal Indeterminacy*] ("Many critical legal scholars . . . urge that law is illegitimate because it is indeterminate.").

[33] *See* Kress, *Legal Indeterminacy*, *supra* note 32, at 295 (positing that "the pervasiveness of easy cases [is] strong support for the thesis that at most there is moderate indeterminacy"). *See generally* Ronald Dworkin, *Hard Cases*, 88 HARV. L. REV. 1057, 1060 (1975) ("I propose, nevertheless, the thesis that judicial decisions in civil cases, even in hard cases . . . characteristically are and should be generated by principle not policy.").

[34] *See* John Stick, *Can Nihilism Be Pragmatic?*, 100 HARV. L. REV. 332, 401 (1986) ("The irrationalist critique [is] that legal reasoning is radically indeterminate (that there are no right answers), and that law cannot be objective.").

[35] *See infra* Part III (proposing a classical model and quantum extension of the Hohfeldian typology).

These measures include "legal entropy," roughly the indeterminacy or "disorder" present in the legal relations in a given legal system, and "legal temperature," a measure of the frequency of change in legal relations in the system.[36] As in the evolution of fields like information theory, economics, linguistics, and political science,[37] these sorts of quantitative measures provide a more robust means for, in Holmesian terms, determining the "prophecies of what courts will do in fact."[38]

Second, the Article posits that while this inherent indeterminacy is indeed inconsistent with classical rationality, it adheres to a *quantum* rationality.[39] Specifically, I build upon Hohfeld's framework and related scholarship to propose a mathematical theory of legal relations that utilizes the formalism of quantum mechanics and quantum computing to describe the inherent indeterminacy of the law.[40] This result shows that law—

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[36] *See infra* Part IV (defining legal entropy and temperature in terms of the mathematical formulation of the Hohfeldian relations); *see also* Sichelman, *Legal Entropy*, *supra* note 21 (providing a formal mathematical theory of legal entropy based on Shannon information entropy).

[37] *See supra* note 6 and accompanying text (noting the application of physics to the social sciences).

[38] Oliver Wendell Holmes, *The Path of the Law*, 10 HARV. L. REV. 457, 460–61 (1897) ("The prophecies of what the courts will do in fact, and nothing more pretentious, are what I mean by the law.").

[39] *See infra* Part III (proposing a quantum formalism for modeling legal relations). Such a quantum approach posits that legal relations, like physical relations, are often "superpositions" of Hohfeldian states that exist in an inherently indeterminate state prior to judgment, analogous to measurement in quantum mechanics. *See id.* Although such inherent indeterminacy is inconsistent with traditional classical legal formalism, like quantum mechanics, it adheres to an analogous quantum logic. *See* KARL SVOZIL, QUANTUM LOGIC (1998) (explaining how quantum mechanics adheres to a nonclassical, nonboolean logical structure); *see also* Peter Mittelstaedt, *Quantum Logic*, *in* PSA: PROCEEDINGS OF THE BIENNIAL MEETING OF THE PHILOSOPHY OF SCIENCE ASSOCIATION 501 (1974) (positing that aspects of Hilbert space used to describe quantum mechanics follows a logical calculus). *Cf.* Christopher L. Kutz, Note, *Just Disagreement: Indeterminacy and Rationality in the Rule of Law*, 103 YALE L.J. 997, 1029 (1994) ("Reason is still secure for the rule of law, as long as the law is properly understood as a forum for argument and criticism and not for determinate conclusions.").

[40] *See infra* Part III (analogizing legal rights prior to judgment to the qubits of quantum computing). Several legal scholars have recognized and assessed the analogy between legal and quantum indeterminacy as well as the effectively quantum mechanisms that may underlie judicial decisionmaking. *See* Jeffery Atik & Valentin Jeutner, *Quantum Computing and Computational Law*, 13 LAW, INNOVATION & TECH. 302, 313–15 (2021) (contending that quantum computers can model indeterminacy in the law); *see also* Christopher Brett Jaeger & Jennifer S. Trueblood, *Thinking Quantum: A New Perspective on Decisionmaking in Law*, 46 FLA. ST. U. L. REV. 733, 784 (2019) (applying the quantum model of decision-making to legal judgments); William H.J. Hubbard, *Quantum Economics, Newtonian Economics, and Law*, 2017 MICH. ST. L. REV. 425, 460–61 (using concepts from quantum theory to explain certain behavioral aspects of judicial decisionmaking); Joseph Blocher, *Schrödinger's Cross: The Quantum Mechanics of the Establishment Clause*, 96 VA. L. REV. IN BRIEF 51, 55 (2010) ("What reality exists before judges render judgment?"); R. George Wright, *Should the Law Reflect the World?: Lessons for Legal Theory from Quantum Mechanics*, 18 FLA. ST. U. L. REV. 855 (1991)

although it may exhibit inherent indeterminacy in interpretation and application—has a rational foundation, or what I term "structure."[41] In turn, the quantum structure of legal entitlements helps to clarify the nature of lawmaking and adjudication, as well as provide an alternative framework for game theoretic models of the law.[42] On a more practical level, because the proposed framework can be formally modeled using quantum bits (qubits) of information,[43] the structure of legal relations can be efficiently encoded as data in a quantum computer—a potentially important result for legal artificial intelligence.[44]

Third, I suggest that the formal, mathematical model of social law proposed here can provide a richer account of scientific law itself.[45] Like the Hohfeldian structure of social law, scientific law exhibits an underlying set of "relations" that describe the "structure" of how those laws govern objects in the material world.[46] This account derives from the relation between so-called modal logic, which provides a logical language to describe what is "necessary" and "possible" in the material world, and deontic logic, which describes what is "obligatory" and "permitted" in the social world.[47] Extending beyond these "flat" logics, I leverage the

(exploring analogies between quantum and legal indeterminacy). *See generally* Lawrence H. Tribe, *The Curvature of Constitutional Space: What Lawyers Can Learn from Modern Physics*, 103 HARV. L. REV. 1, 2 (1989) (reflecting that the "metaphors and intuitions that guide physicists can enrich our comprehension of social and legal issues"); Massimiliano Ferrara & Angelo Roberto Gagliotti, *Legal Values and Legal Entropy: A Suggested Mathematical Model*, 3 INT'L J. MATH. MODELS & METHODS IN APPLIED SCI. 490 (2012) (purporting to develop a "mathematical" approach to the law and introducing a variety of connected symbols but ultimately providing a metaphorical treatment). However, no scholar has offered a formalization of the analogy, which, importantly, allows for the application of the mathematical formalism of quantum physics to model and describe social law. *See also supra* note 5 and accompanying text.

[41] *See infra* Part V (contrasting the "structure" and "content" of the law).

[42] *See infra* Part V (explaining the relationship between quantum game theory and the law); s*ee also* Ted M. Sichelman, *Quantum Game Theory and Coordination in Intellectual Property* (Nov. 19, 2015) (Working Paper), https://ssrn.com/abstract=1656625.

[43] *See infra* Part III (extending the notion of a Hohfeldian relation to a probabilistic one using qubits). *See generally* Sichelman, *Legal Entropy*, *supra* note 21, at 264 (describing legal relations as qubits in the context of legal entropy).

[44] *See* Atik & Jeutner, *supra* note 40, at 321–22 (explaining the importance of quantum computing for legal artificial intelligence).

[45] *See infra* Part VI (applying the Hohfeldian model of legal relations to physical relations). *See generally supra* note 4 (noting references that examine the relationship between natural and social law).

[46] *See infra* Part VI (explaining that scientific laws extend beyond models of physical phenomena to laws that govern physical phenomena).

[47] *See infra* Part VI (positing that fundamental physical constituents can be viewed via deontic logic under the assumption that such constituents are "actors" that never violate the governing rules).

mathematical model proposed here to offer a multi-ordered, modular description of scientific law that can help to explain phenomena seemingly outside of current explanation, such as quantum measurement.[48]

The Article proceeds as follows: Part II briefly describes Hohfeld's framework. Part III offers a syntactic formalization of the Hohfeldian approach that is similar to previous logical formalizations. Part IV begins by proposing a "classical" mathematical model of the Hohfeldian relations in which there is no indeterminacy. Part IV then offers a "quantum" model of the Hohfeldian relations in order to incorporate inherent indeterminacy. Using these mathematical models, Part V introduces several measures—including entropy, temperature, information content, and modularity—which can be used to provide precise quantitative descriptions of legal systems. Part VI then examines applications of the proposed model to theories of adjudication, legal artificial intelligence, and game theory and the law.

Part VII applies the proposed model of legal relations back to scientific laws and theories, positing that contrary to prior models, scientific laws—like social laws—occupy different "orders." First-order scientific laws describe the behavior of idealized objects (such as point particles, strings, and fields), while second-order scientific laws concern certain forms of measurement (particularly, quantum measurement) as well as the potential creation, termination, and change of scientific laws.[49]

The Article concludes by briefly discussing several implications of the proposed model.

I. HOHFELD'S LEGAL FRAMEWORK

A. Hohfeld's Typology of Legal Rights & Obligations

There is no consensus on the precise meaning of the term "legal right." Hohfeld showed that the ambiguity of this term stemmed from using it to denote multiple, discrete legal concepts.[50] As an alternative, Hohfeld proposed a typology of eight precise, atomic "jural relations," which he

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[48] *See infra* Part VI (applying the probabilistic Hohfeldian model of legal relations offered herein to the problem of quantum measurement).

[49] *See infra* Conclusion (contending that the Hohfeldian framework can help explain the ontological nature and origins of scientific laws).

[50] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 30 ("[T]he term 'rights' tends to be used indiscriminately to cover what in a given case may be a privilege, a power, or an immunity, rather than a right in the strictest sense.").

argued could be used to describe all "legal problems."[51] These jural relations are *right*, *duty*, *privilege*, *no-right*, *power*, *liability*, *immunity*, and *disability*.[52] The *right/duty/privilege/no-right* set of jural relations ("first-order relations") concern the constraints (or lack thereof) on the behavior ("actions") of persons subject to a given legal system ("legal actors").[53] "Higher order" jural relations of *power*, *liability*, *immunity*, and *disability* concern the creation, termination, or change of other, usually first-order, jural relations.[54]

A simple example of a first-order relation is that of trespass.[55] Assume that legal actor *A* owns land *L* (see Fig. 1) and *B* is some other legal actor that has no ownership interest in *L*.[56] As briefly mentioned earlier, one standard incident of property ownership is the "right to exclude," that is, the right to prevent trespassers (under certain circumstances) from entering the owner's land.[57] If *B* has no valid excuse to enter *L*, in Hohfeldian terms, *A* has a "right"— specifically, vis-à-vis *B*—that *B* not enter *L*.[58] (Because a Hohfeldian *right* is a precise form legal right, I typically refer to it as a "strict-right" in the following discussion.)[59] *B* has a "correlative" Hohfeldian "duty"—vis-à-vis *A*—*not* to enter *L*.[60] Thus, a *strict-right* and a *duty* are Hohfeldian correlatives.[61] In other words, if (and only if) *A* has a *strict-right* via *B*, then *B* has a *duty* via *A*—of course, with respect to the

[51] *See id.* at 28–30 ("One of the greatest hindrances to the clear understanding, the incisive statement, and the true solution of legal problems frequently arises from the express or tacit assumption that all legal relations may be reduced to 'rights' and 'duties' . . . .").

[52] *See id.* at 30 (introducing eight jural relations related to one another via "opposites" and "correlatives").

[53] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 60–61 & n.61 (describing the relationship between first-order and higher-order Hohfeldian relations); *see also infra* Part II.A (explaining how a Hohfeldian proposition comprises atomic Hohfeldian relations, actors, and actions).

[54] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 45–46 (explaining how operative facts, which effectuate a power, can create or extinguish legal relations).

[55] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 52 n.57 (discussing trespass in the context of Hohfeldian rights, duties, privileges, and no-rights).

[56] *See* Honoré, *supra* note 25 (explaining that property ownership typically includes a right against trespass, a privilege to use, a power to alienate, a power to abandon, a power to destroy, and others).

[57] *See id.* (discussing trespass as an incident as property ownership).

[58] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 32 ("In other words, if X has a right against Y that he shall stay off the former's land . . . .").

[59] *See id.* at 30 (noting that "right" in the Hohfeldian schema is "a right in the strictest sense").

[60] *See id.* at 32 ("[T]he correlative (and equivalent) is that Y is under a duty toward X to stay off the place.").

[61] *See id.* at 30 (noting that strict-right and duty are correlatives, along with privilege and no-right).

same underlying action (here, not entering *L*).[62] Just as *A* having a *strict-right* implies that *B* has a *duty*, if *A* has no *right* (termed a *no-right* by Hohfeld), then *B* has no *duty* (termed a *privilege* by Hohfeld).[63] So *strict-right* and *duty* are Hohfeldian correlatives, and so are *no-right* and *privilege*.[64] From another perspective, *strict-right* and *no-right* are Hohfeldian "opposites," and so are *duty* and *privilege*.[65]

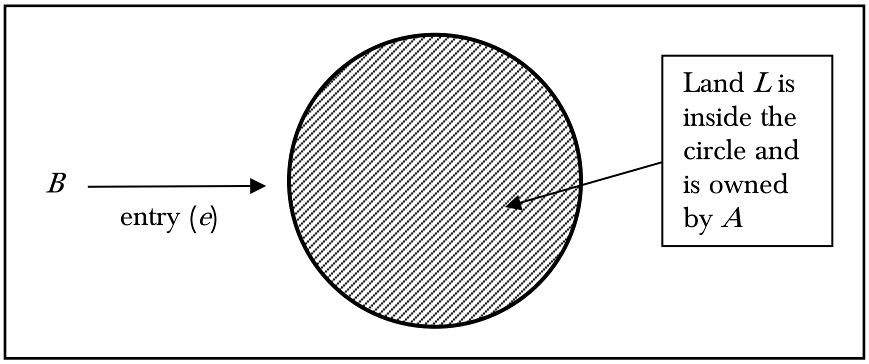

Fig. 1. Potential Entry onto *A*'s Land *L* by Non-owner *B*

In other words, the law of property—by way of *A*'s "ownership" of *L*—provides *A* with a *strict-right* to keep *B* from entering *L*, and creates a correlative *duty* in *B* not to enter *L*.[66] On the other hand, if *A* lacks ownership in a finite region outside of *L* (call it *M*), *A* holds *no strict-right* to restrain *B* from entering *M*.[67] In Hohfeldian terms, *A* has a "no-right" that *B* not enter *M*, and *B* has a correlative "privilege" to enter *M*.[68] (See Fig. 2.)

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[62] *See id.* at 31 ("[E]ven those who use the word and the conception 'right' in the broadest possible way are accustomed to thinking of 'duty' as the invariable correlative.").

[63] *See id.* at 32 ("[A] privilege is the opposite of a duty, and the correlative of a 'no-right.'").

[64] *See id.* at 30 (providing a table of "correlatives" of each of the Hohfeldian relations).

[65] *See id.* (providing a table of "opposites" of each of the Hohfeldian relations). Logically, jural opposites are simply negations of one another. More generally, the table of jural "opposites" in Hohfeld can be aligned with the deontic square of opposition, implying that privilege and duty are logical "contradictories." Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 22 n.16.

[66] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 32 ("[I]f X has a right against Y that he shall stay off the former's land, the correlative (and equivalent) is that Y is under a duty toward X to stay off the place.").

[67] *See id.* at 32–33 (explaining no-rights in the context of property ownership).

[68] *See id.* at 33 ("[t]he correlative of X's privilege of entering [X's land] himself is manifestly Y's 'no-right' that X shall not enter."). Note that in the Hohfeldian scheme *A*'s *no-right* that *B*

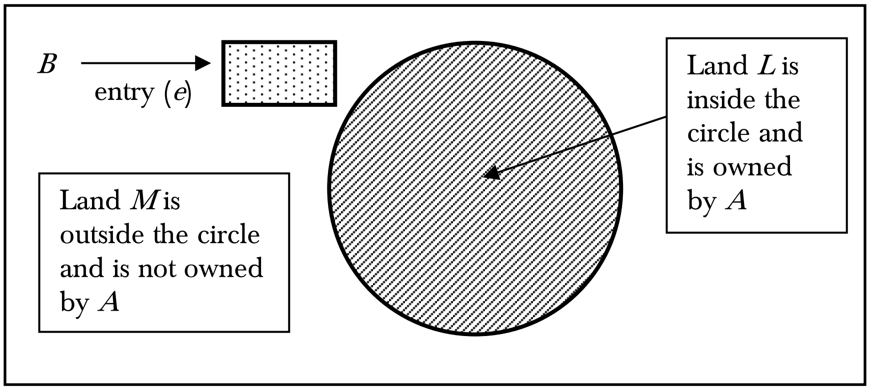


Fig. 2. Potential Entry onto Land *M* that is Not Owned by *A* by Non-owner *B*

In sum, if *A* has a *strict-right*, *B* has a *duty*; if *A* has *no-right*, then *B* has a *privilege*.[69] So *strict-right* and *duty* are Hohfeldian correlatives, and so are *no-right* and *privilege*.[70] From another perspective, *strict-right* and *no-right* are Hohfeldian "opposites," and so are *duty* and *privilege*.[71] The correlativity and oppositeness of the first-order Hohfeldian relations are summarized in Figure 3 below.[72]

not enter M is correlative to *B*'s *privilege* (vis-à-vis *A*) *to enter M*. On the other hand, *A*'s *strict-right* that B *not enter L* is correlative to *B*'s *duty* (vis-à-vis *A*) *not to enter L*. Thus, the "tenor" of the action is asymmetric with respect to *no-rights* and correlative *privileges*, but not with respect to *strict-rights* and corresponding *duties* (Hohfeld). *See id.* at 32–33 (explaining the notion of "tenor").

[69] *See id.* at 30–33 (presenting and explaining the table of Hohfeldian correlatives).

[70] *See id.*

[71] *See id.*

[72] *See id.*

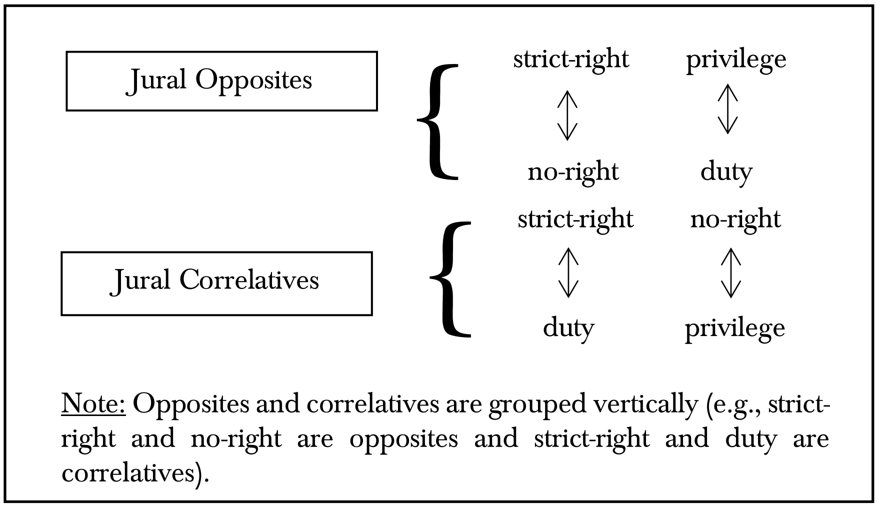


Fig. 3. First-order "Jural Opposites" & "Jural Correlatives"[73]

B. Hohfeld's Typology of Powers & Immunities

Hohfeldian *powers*, *immunities*, *liabilities*, and *disabilities* are "higher-order" jural relations that govern the ability (or lack thereof) of legal actors to create, change, or extinguish "lower-order" legal relations.[74] For example, suppose *A* (our landowner) and another legal actor *C* enter into a contract whereby *A* transfers ownership of his land *L* to *C*.[75] As soon as the ownership of *L* has transferred to *C*, *A*'s then-existing *strict-right* to prevent *B* from entering *L* transforms into a *no-right*—that is, once *A* is no longer the owner of *L*, *A* has "no right" to keep *B* off *L*.[76] In Hohfeldian terms, *A* has the *power* (vis-à-vis *B*) to transform *A*'s *strict-right* (again, vis-à-vis *B*) that *B* not enter *L* into a *no-right* by transferring ownership of *L* to a third party, such as *C*.[77] That *B* is subject to *A*'s *power*

[73] *See id.* at 30. The legal relations are presented here in the same order as presented in Hohfeld's original article. *See id.*

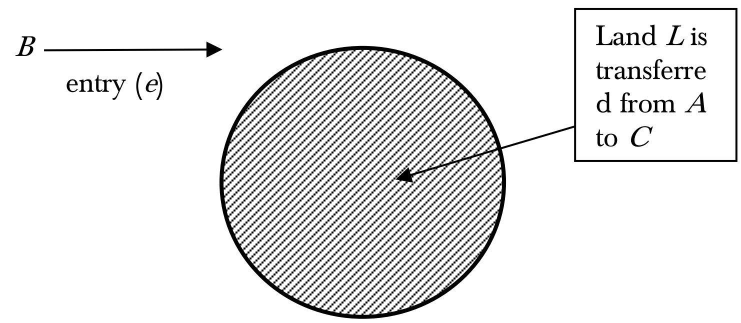
[74] *See id.* at 44–47 (introducing the notion of legal powers, immunities, liabilities, and disabilities); Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 21 n.14 (describing first- and second-order Hohfeldian relations).

[75] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 45 (discussing the power to alienate property).

[76] Related, *C*'s then-existing *no-right* that *B* not enter *L* transforms into a *strict-right*. *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 52 (describing the nature of the change in Hohfeldian relations upon the transfer of property).

[77] *See id.* ("[T]he acquiror has a privilege to read the book; a strict-right to keep others from reading it; powers to transfer, abandon, or destroy it; and so forth.").

is characterized in Hohfeldian terms by stating that *B* has a correlative *liability*.[78] (See Fig. 4.)



Once A sells his land to C, A no longer has a strict-right to keep B from entering L; instead, A now has a no-right. The change of A's relation (vis-à-vis B) from strict-right to no-right is effectuated by A's exercise of a legal power transferring ownership of L.

Fig. 4. Transfer of L from A to C and Change of A's Legal Relations

Once *A* sells his land to *C*, *A* no longer has a *strict-right* to keep *B* from entering *L*; instead, *A* now has a *no-right*. The change of *A*'s relation (vis-à-vis *B*) from *strict-right* to *no-right* is effectuated by *A*'s exercise of a legal power transferring ownership of *L*.

If a legal actor lacks the power to alter, create, or terminate a given jural relation, then in Hohfeldian terms, that actor has a *disability*.[79] Suppose, for instance, that *D* is simply a long-term guest of the actual landowner, now *C*.[80] In this instance, *D* may have a *strict-right* to prevent *B* from entering *L* (i.e., to keep trespassers away), but still no authority to sell the land.[81] Suppose the guest *D* attempts to transfer ownership via a

[78] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 44 (positing that power and liability are correlatives).

[79] *See id.* at 55 (describing immunities and disabilities).

[80] *See id.* ("For Y is under a disability (i.e., has no power) so far as shifting the legal interest either to himself or to a third party is concerned; and what is true of Y applies similarly to every one else who has not by virtue of special operative facts acquired a power to alienate X's property.").

[81] *Cf.* People v. Wagner, 104 Mich. App. 169, 175 (1981) (holding that the defendant had legitimate expectation of privacy because he had moved into his girlfriend's townhouse, had been there an indefinite time, and kept his clothes there).

supposed "deed of sale" back to the previous owner *A* without *C*'s authorization, and *A* knows *C* is the actual owner of *L*.[82] In this event, there will be no sale, because the guest *D* lacks the *power* (i.e., has a Hohfeldian *disability*) to transfer ownership rights in *L* to the previous owner *A*.[83] Correlatively, in Hohfeldian terms, the actual owner *C* is "immune" (i.e., is subject to a Hohfeldian *immunity)* from any attempt of the guest *D* to transfer *L* to the previous owner *A*.[84] That is, the actual owner *C* is immune from the legal effects of the underlying primary actions that the guest *D* might take to transfer *L* (e.g., *D*'s signing a supposed "deed of sale" to the previous owner *A*).[85]

To recap, if a first legal actor *X* has a *power* vis-à-vis a second legal actor *Y* to change, create, or terminate a lower-order legal relation, then *Y* has a correlative *liability* with respect to *X*'s exercise of the *power*, such that the lower-order relation is changed, created, or terminated.[86] On the other hand, if *X* has no *power*, then *X* is *disabled* vis-à-vis *Y* from affecting the lower-order legal relation, and *Y* is correlatively *immune* from *X*'s attempt to change the lower-order relation.[87] The correlativity and oppositeness of the higher-order Hohfeldian relations are summarized in Figure 5.[88]

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[82] In the event *A* does not know *C* is the actual owner, if *D* somehow comes into ownership of the property, then the conveyance may be effective under the "estoppel by deed" doctrine. *See* 28 AM. JUR. 2D *Estoppel and Waiver* § 8 (2024) ("[I]f the party claiming the benefit of the estoppel has not been misled by the other party's deed, or recital therein, there is no estoppel.").

[83] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 55–56 (noting that a general third party has a "disability" to alienate the land of another).

[84] *See id.* at 56–57 (explaining the concept of "immunities").

[85] *See id.* I have described in the text only the *immunity-disability* relation between *C* (the present owner) and *D* (the guest and would-be seller). There is a separate *immunity-disability* relationship between *A* (the former owner) and *D*; namely, *A* is *immune* from any attempt from *D* to transfer *L* to him. That is, despite *A* and *D*'s best efforts to effectuate a transfer, *A* will never come into ownership of *L*. Finally, *B* (the would-be trespasser) is *immune* from any attempt of *D*, by a supposed transfer of *L* to *A*, to alter *B*'s *duty* (which is vis-à-vis *C*) not to enter *L*. In general, the law is concerned with powers and duties, not immunities and privileges, because the latter categories refer to those actions that have no legal effect (immunities) or do not violate the law (privileges). To the extent those actions are not exceptions from a background legal rule, there is typically little to no need for the law to expressly deal with them. *Cf.* Henry E. Smith, *Exclusion Versus Governance: Two Strategies for Delineating Property Rights*, 31 J. LEGAL STUD. 453, 473 (2002) ("The theoretical limit, achievable only under conditions of zero transaction costs, would be one in which every potential Hohfeldian legal relation (right/duty, privilege/no right, and so on) is specified between every pair of members of the society.").

[86] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 44–47 (explaining the relationship between a power and liability).

[87] *See id.* at 55–57 (explaining the relationship between immunities and disabilities).

[88] *See id.* at 30 (presenting a table of correlatives and opposites for powers, liabilities, immunities, and disabilities).

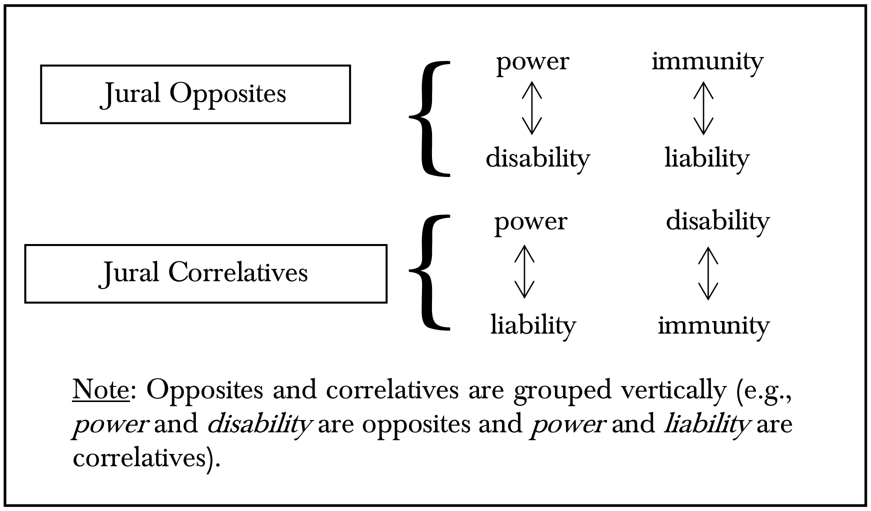

Fig. 5. Higher-order "Jural Opposites" & "Jural Correlatives"[89]

There is an important wrinkle to Hohfeldian powers. Most *powers* are "second-order" in the sense that they affect a first-order relation.[90] For example, the *power* of a landowner *A* to change a first-order *strict-right* that a third party *B* not enter *L* into a first-order *no-right* through the sale of *L* to a purchaser *C* is a second-order *power*.[91] In general, an *n*th-order *power* affects an *(n-1)*th-order legal relation.[92] For instance, the State's "power" of eminent domain—which allows the State to take ownership of a private individual's property (typically, in exchange for appropriate monetary compensation)—includes within its scope a third-order Hohfeldian *power* to change the owner's second-order *power* to transfer the property into a second-order *disability*.[93] Numerous examples of other third- and higher-order *powers* appear in contract and constitutional law.[94]

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[89] The fundamental relations are presented in the same order as in Hohfeld's original article. *See id.*

[90] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 61 nn.61, 63 (discussing second- and higher-order powers in the context of Hohfeld's original work).

[91] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 55–57 (discussing powers in the context of the alienation of property).

[92] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 61 nn.61, 63 (explaining that higher-powers alter lower-order powers).

[93] *See id.* at 61 ("For instance, the State may use a higher-order power to abridge private parties' lower-order contractual powers.").

[94] *See id.* ("The constitutional amendment power is an example of a potentially even higher-order power (e.g., the constitution may abridge a state's higher-order power to abridge private parties' lower-order powers)."). The foregoing description of Hohfeld's framework is brief, but covers the material essential to the remaining sections. Readers desiring more background

II. LOGICALLY FORMALIZING THE HOHFELDIAN FRAMEWORK

In the latter half of the 20th century, logicians and others developed a burgeoning literature of highly formal approaches to Hohfeld, forming close linkages to the field of deontic logic.[95] This Part draws upon this literature to offer a brief, logical account of Hohfeldian relations that will prove helpful in formulating the mathematical account proposed in the next Part of the Article.[96]

A. Symbolizing Classical Rights & Obligations

The relationships among the first-order Hohfeldian relations may be described in a more compact notation.[97] Recall the example regarding the would-be trespasser *B* on *A*'s land *L*.[98] In this instance, *A* has a *strict-right* that *B* not enter *L*, which is equivalent to stating that *B* has a *duty* not to enter *L*.[99] On the other hand, *A* has a *no-right* that *B* not enter *M* (a small area outside *L* that is not owned by *A*).[100] Equivalently, *B* has a *privilege* of entering *M*.[101] One table (with arrows) may be used to display the Hohfeldian "oppositeness" and "correlativity" of these first-order legal relations.[102] (See Fig. 6.)

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should consult the key excerpts of Hohfeld's landmark 1913 article setting forth his theory as well as useful secondary works that more fully explain Hohfeld's typology. *See, e.g.*, Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 20–25, 28–49; *see also* Arthur L. Corbin, *Legal Analysis and Terminology,* 29 YALE L.J. 163 (1919); Arthur L. Corbin, *Jural Relations and Their Classification,* 30 YALE L.J. 226 (1920); John Finnis, *Some Professorial Fallacies About Rights*, 4 ADEL. L. REV. 377 (1971); Singer, *supra* note 13; Schlag, *supra* note 7.

[95] *See supra* note 12 (listing references).

[96] *See infra* Part III (offering a mathematical formalization of the Hohfeldian relations).

[97] *See* David Makinson, *On the Formal Representation of Rights Relations: Remarks on the Work of Stig Kanger and Lars Lindahl*, 15 J. PHIL. LOGIC 403, 403–25 (1986) (proposing and analyzing compact formalizations of the Hohfeldian relations).

[98] *See supra* notes 55–68 and accompanying text (describing the example of a trespasser in Hohfeldian terms).

[99] *See supra* notes 55–68 and accompanying text.

[100] *See supra* notes 55–68 and accompanying text.

[101] *See supra* notes 55–68 and accompanying text.

[102] *Cf. supra* Fig. 3 (presenting Hohfeld's original two tables of the first-order jural relations). In what follows, I use the term "legal relations" interchangeably with the Hohfeldian "jural relations."

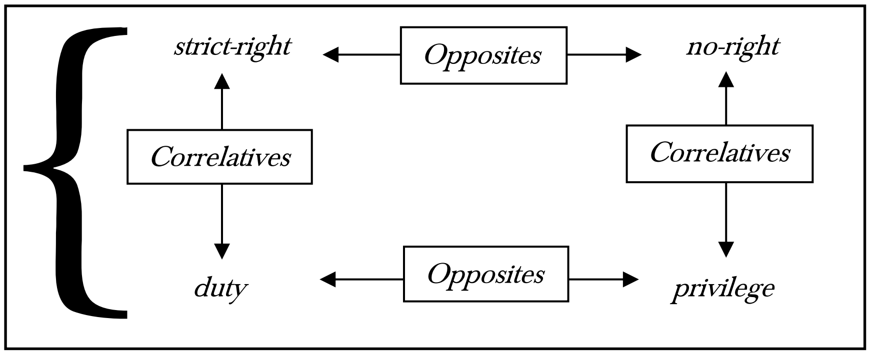

Fig. 6. A Single "Square" of the First-Order Relations

The first step in formalizing the Hohfeldian relations is to replace them with appropriate symbols.[103] Let a *strict-right* be symbolized by the letter *r*. In this event, a *no-right* is just $\sim r$ (where "~" indicates negation).[104] Let *duty* be symbolized by r^c (where "c" indicates a "correlative").[105] Finally, since *privilege* is the negation of *duty*, it is symbolized by $\sim r^c$.[106] These relationships are summarized in Figure 7.

[103] *See supra* note 12 (listing references providing similar formalizations).

[104] Hohfeld termed a *strict-right* and *no-right* as "opposites." As Hohfeld appeared to recognize, in more precise logical terms, $\sim r$ is the negation (or absence) of *r*. *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 32 ("privilege is the mere negation of a *duty*"); *see also* Giovanni Sartor, *Fundamental Legal Concepts: A Formal and Teleological Characterisation*, 14 A.I. & L. 101 (2006) (offering a logical formalization of the Hohfeldian relations).

[105] *See generally* A.K.W. Halpin, *Hohfeld's Conceptions: From Eight to Two*, 44 CAMBRIDGE L.J. 435, 456–57 (1985) (proposing a reduction in the Hohfeldian relations simply to right and duty).

[106] *See id.* (defining a privilege in terms of a duty and logical connectives). As this Article shows in the next Part, the Hohfeldian relations may be reduced simply to any one relation, whereby all of the others can be derived via logical and mathematical operations. *See infra* Part III. In essence, one can posit that "all law is right," "all law is power," or "all law is duty." In Hohfeldian terms, all of the statements are equivalent. Of course, one's starting point (e.g., right, privilege, or power) may in practice affect how the law is ultimately formulated. *Cf.* Sichelman, *Legal Entropy*, *supra* note 21, at 9 ("[O]ne can imagine the complete Hohfeldian state space as a collection of a multitude of vectors and tensors corresponding to every possible action and states of the world affected by law (and a complement space of all of those actions and states not so affected).").

First-Order Legal Relations	r	$\sim r$
	r^c	$\sim r^c$

Here, r is a *strict-right*, r^c is a *duty*, $\sim r$ is a *no-right*, and $\sim r^c$ is a *privilege*.

Fig. 7. A Single "Square" of First-Order Legal Relations

Using these abbreviations, *A*'s *strict-right* vis-à-vis *B* that *B* not enter *L* may be written as:

$$A_r B \text{ } (B \text{ not enter } L) \quad (1)$$

This expression is equivalent to:

$$B_{r^c} A \text{ } (B \text{ not enter } L) \quad (2)$$

(i.e., *B* has a *duty* via-a-vis *A* not to enter *L*).[107]

In general, all forms of "complete," classical first-order legal relations—in other words, a *legal proposition* such as (1) and (2)—contain three elements.[108] First, there must be at least two legal actors to which the relation pertains.[109] These actors may be real persons or artificial legal entities, such as corporations, partnerships, and even the government ("the State").[110] For instance, in statements (1) and (2) above, *A* and *B* are the

[107] *See* Anthony Dickey, *Hohfeld's Debt to Salmond*, 10 U.W. AUSTL. L. REV. 59, 59–63 (1971) (examining the historical and analytical lineage of the relationship between strict-rights and duties).

[108] *See* Makinson, *supra* note 97, at 404–06 (describing the three elements common to legal relations in Kanger's formalization of the Hohfeldian typology).

[109] *See id.* at 405–06 (describing Kanger's "simple types of rights relation with respect to two parties").

[110] Although Hohfeld denied that composite entities such as a corporation were legal actors per se, and this remains a thorny jurisprudential concern, there is little question that such composite actors may occupy one of the two Hohfeldian "actor" positions in a legal proposition and yield coherent, meaningful statements. *See* Bryant Smith, *Legal Personality*, 37 YALE L.J. 283, 295 (1928) (offering an early critique of Hohfeld's view of corporations and remarking that "if the sovereign power confers legal personality upon a ship, or an idol, or upon an abstraction, such as one of the functional aspects of an individual or of an organized group, such ship or idol or functional aspect ipso facto is a party to; legal relations").

two legal actors of concern.[111] (For simplicity, I assume herein that a legal proposition *only* concerns two actors.)[112] Second, there must be a "specific" first-order legal relation (i.e., r, $\sim r$, r^c, or $\sim r^c$) between the two actors.[113] For first-order classical relations, either the two actors have a *strict-right/duty* relation or a *no-right/privilege* relation.[114] That is, one actor (e.g., B) either owes a *duty* to A or not.[115] Third, the specific relation between the two actors will concern a specific "state of affairs" of the world.[116] Typically, the state of affairs is an action that the actor with a *duty* must engage in (a "positive" *duty*), must *not* engage in (a "negative" *duty*), may engage in (a "positive" *privilege*), or may *not* engage in (a "negative" *privilege*).[117] More generally, legal propositions may concern any factual state of the world, at some particular instant in time, over a continuous period of time, or throughout multiple, disconnected periods.[118]

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[111] *See* Makinson, *supra* note 97, at 404–06 (noting that Kanger formalized actors as an ordered pair, $\binom{x}{y}$).

[112] Hohfeld argued that legal relations at the most fundamental level could inhere only between two persons. *See* Corbin, *Legal Analysis and Terminology*, *supra* note 94, at 165 ("The term 'legal relation' should always be used with reference to two persons, neither more nor less."). However, his contention remains an open question. For instance, one's current self could owe a duty to one's future self, enforced via the State. Another possibility is that certain rights are held jointly and indivisibly by multiple individuals. *Cf.* JAMES E. PENNER, THE IDEA OF PROPERTY IN LAW 25–26 (1997) ("To understand rights *in rem* we must . . . discard Hohfeld's dogma that rights are always relations between two persons."). The assumption here that there are only two actors is not critical, as it would be relatively straightforward to extend the formalism to relations between multiple actors.

[113] *See* Sartor, *supra* note 104, at 110–14 (describing various types of Hohfeldian relations in the context of legal propositions).

[114] *See supra* notes 50–72 and accompanying text (explaining the right/duty and no-right/privilege relations).

[115] *See supra* notes 50–72 and accompanying text (noting the binary nature of the Hohfeldian relations).

[116] *See* Makinson, *supra* note 97, at 407 ("In general, a statement saying that X performs a certain kind of action is represented by means of the do operator as saying that Y sees to it that a certain state of affairs holds . . . ."). Although the Hohfeldian schema is typically viewed as concerns legal restraints on human behavior, of course, states of affairs may extend well beyond human action. For instance, it is altogether logically possible that a legal actor may have an obligation to "see to it" that it not rain tomorrow, even though that actor has no ability to affect whether it will indeed rain.

[117] *See* Sartor, *supra* note 104, at 103–04 (describing "positive" and "negative" actions with respect to deontic and Hohfeldian relations).

[118] *See* BO R. MEINERTSEN, METAPHYSICS OF STATES OF AFFAIRS: TRUTHMAKING, UNIVERSALS, AND A FAREWELL TO BRADLEY'S REGRESS 6 (2019) (describing how time plays a role in states of affairs).

As such, the general form of any first-order legal proposition is as follows:[119]

[Actor #1][legal relation][Actor #2] (a state of affairs that the legal relation concerns) (3)

Let "Actor #1" be X, the first-order specific legal relation be j_1, Actor #2 be Y, and the state of affairs, S; then all first-order classical legal propositions, J_1, take the following form:[120]

$$J_1 = X_{j_1} Y(S) \tag{4}$$

In view of the "correlativity" principle of the Hohfeldian framework, by convention, it is possible to orient this form (of J_1) always in "*strict-right* notation," meaning that X is always the legal actor that holds a *strict-right* (or not) and Y is always the legal actor that is subject to a *duty* (or not).[121] In this case:

$$j_1 = r \text{ or } {\sim}r \tag{5}$$

Thus, in *strict-right* notation, any first-order legal proposition must take the form:[122]

$$J_1 = X_{j_1} Y(S) \text{ where } j_1 = r \text{ or } {\sim}r \tag{6}$$

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[119] Here, "first-order legal proposition" refers here to a specific legal statement that expresses the legal strict-rights (and corresponding duties) between two legal actors with respect to a specific state of affairs obtaining or not, typically premised upon an action undertaken (or not) by one of the actors. As discussed in Part V.A, much of what the law concerns is determining legal propositions from various legal sources, such as constitutions, statutes, and cases in view of policies and principles that guide legal interpretation. *See infra* Part V.A. Once a specific legal proposition is formulated, one applies the law to the facts. *See* Henry P. Monaghan, *Constitutional Fact Review*, 85 COLUM. L. REV. 229, 236 (1985) ("If all legal propositions could be formulated in great detail, [law application] would be rather mechanical and require no distinctive consideration.").

[120] Here, the terms "legal" and "jural" can used interchangeably. I use "j" instead of the letter "l" to denote legal relations and propositions for two reasons: one, "j" is easier to distinguish (particularly in lower-case) when it is adjacent to the number "1"; two, it follows Hohfeld's original terminology of "jural relation." Additionally, the variables "X" and "Y" for the actors are meant to convey that they are general and may take as an argument any specific actor (e.g., *A*, *B*, *C*, etc.).

[121] *See* Makinson, *supra* note 97, at 405 (describing Kanger's approach, which orients legal actors in an ordered pair with respect to underlying Hohfeldian duties or the lack thereof).

[122] Following deontic logic's focus on obligations, many Hohfeldian formalizations use duty notation, *see, e.g.*, *id.*, but using a strict-right notation more effectively aligns the formalism with a power notation for second- and higher-order relations, given the conceptual similarity between powers and rights. *See infra* note 157 and accompanying text.

The application of any precise legal rule concerning a first-order legal relation involving two particular legal actors and a specific state of affairs may be expressed in the form of (6).[123]

B. Symbolizing Classical Powers & Immunities

Recall from the earlier discussion that a Hohfeldian *power* changes, terminates, or creates lower-order legal relations, and that an actor who is subject to a *power* holds a correlative Hohfeldian *liability*.[124] If a first actor lacks a *power* with respect to a second actor, the first actor holds a Hohfeldian *disability*, and the second actor holds a correlative Hohfeldian *immunity*.[125] As shown in Figure 8, if a *power* is designated as p, then a *liability* is simply the correlative of a *power*, p^c.[126] A *disability* is the negation (or absence) of a *power*, $\sim p$, which is the correlative of an *immunity*, $\sim p^c$.[127]

Higher-Order Legal Relations
$$\left\{ \begin{array}{cc} p & \sim p \\ p^c & \sim p^c \end{array} \right.$$

Here, p is a power, p^c is a liability, $\sim p$ is a disability, and $\sim p^c$ is an immunity.

Fig. 8. A Single "Square" of the Higher-Order Relations

[123] By "precise," I mean that the legal rule—when applied to a state of affairs (i.e., a set of facts)—yields a unique answer (i.e., whether a *right* or *no-right* exists between the given legal actors). I relax this assumption in Section 4. Additionally, by "legal rule," I mean to include traditional legal rules (e.g., "The speed limit is 55 mph.") as a well as legal "standards" (e.g., "Drive at a speed that is reasonable, taking into account road conditions, time of day, weather, the number of vehicles on the road, etc."). *Cf.* Duncan Kennedy, *Form and Substance in Private Law Adjudication*, 89 HARV. L. REV. 1685 (1976).

[124] Here, I focus on the Hohfeldian powers that change legal relations. Powers that create or terminate legal relations are quite different from transformative powers, and I discuss them briefly below. *See infra* note 158.

[125] *See supra* notes 74–94 and accompanying text (explaining powers, liabilities, immunities, and disabilities).

[126] *See supra* notes 74–94 (noting that powers and liabilities are Hohfeldian correlatives).

[127] *See supra* notes 74–94 (noting that the absence or negation of power is a disability and the correlative of a disability is an immunity).

Using these abbreviations, and following the earlier hypothetical, landowner *A*'s second-order *power* to change his *strict-right* that a would-be trespasser *B* not enter *A*'s land *L* into a *no-right* by A's transferring its legal interests in *L* to some third party *C* may be written as follows:[128]

$$A_pB\ (A\ transfers\ L\ to\ C)[A_rB\ (B\ not\ enter\ L)] \qquad (7)$$

In (7), the relation on the left-hand side (i.e., "A_pB *(A transfers L to C)*")[129] is the second-order legal proposition proper,[130] which concerns the first-order proposition on the right-hand side (i.e., "A_rB *(B not enter L)*").[131] Here, the second-order relation is a *power*, symbolized by the subscript *p*

[128] In contrast to formalizations of strict-rights and duties, relatively few scholars have attempted to formalize Hohfeldian powers. Nearly all of these scholars have employed a hybrid approach that relies heavily on "if-then," first-order logic to model powers, rather than a pure "structural," operator-focused approach that mirrors the formalization of the first-order relations. *See, e.g.*, Andrew J.I. Jones & Marek Sergot, A FORMAL CHARACTERISATION OF INSTITUTIONALISED POWER, 4 J. IGPL 427 (1995). Yet, first-order approaches fail to reflect the modular, ordered use of powers and rights in the law, which affords a substantial reduction in information costs in the legal system. Indeed, although "if-then" approaches are certainly suitable to describe powers, they are suitable to describe rights and duties as well, so it is arguably incongruous for the formalization literature to adopt one approach for powers, yet another for rights. *Cf.* T.J.M. Bench-Capon, *Deep Models, Normative Reasoning and Legal Expert Systems*, *in* PROCEEDINGS OF THE 2ND INTERNATIONAL CONFERENCE ON AI AND LAW 37 (1989) (describing the benefits of a "shallow" model of law that relies on first-order logic rather than a "deep" model of law that uses deontic logic).

[129] Here, I have condensed "*A* transfers its legal interests in *L* to *C*" to "*A* transfers *L* to *C*." Although Hohfeld himself criticized such terminology—because it potentially conflates the legal interests in land with the land itself—the meaning in this context is amply clear. *See* Hohfeld, *Fundamental Legal Conceptions, supra* note 14, at 21–23 ("A . . . reason for the tendency to confuse or blend non-legal and legal conceptions consists in the ambiguity and looseness of our legal terminology. The word 'property' furnishes a striking example."). Indeed, to require lawyers to use precise terminology in situations in which there is no chance of ambiguity would impose substantial but unnecessary information costs. *See* Ted M. Sichelman, *Very Tight 'Bundles of Sticks': Hohfeld's Complex Relations*, *in* WESLEY HOHFELD A CENTURY LATER: EDITED WORK, SELECT PERSONAL PAPERS, AND ORIGINAL COMMENTARIES 345 (Shyam Balganesh, Ted Sichelman & Henry Smith eds., 2022) (positing that imprecision in legal language is generally justified when the meaning is clear). As such, unlike Hohfeld, I do not perceive a need for linguistic nicety in every legal proposition.

[130] The event "*A* transfers *L* to *C*" is a shorthand for those acts that the law recognizes as effectuating a power, for instance, the signing of a contract, the enactment of legislation, and so forth. Recognition of second- and higher-order legal acts is more precisely achieved through a "rule of recognition" (typically, sets of related rules) that deem certain actions to have legal effect (or not). *See* HART, *supra* note 1, at 94–110 (describing the rule of recognition).

[131] In general, a nth-order relation will always concern an (n-1)th-order relation. *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 61–62 nn.62–64 and accompanying text (explaining the orders of legal relations).

between legal actors *A* and *B*.[132] *A* holds the *power,* and *B* is subject to it (i.e., *B* holds a *liability*).[133] *A*'s exercise of the *power* is effectuated when the state of affairs "*A* transfers *L* to *C*" occurs.[134] Immediately upon this occurrence, there is a change in the first-order relation from the *strict-right* relation "A_rB *(B not enter L)*" to the *no-right* relation "$A_{\sim r}B$ *(B not enter L)*."[135]

In general, all forms of second-order legal propositions will concern four elements.[136] First, there is a first-order legal relation (concerning a first-order legal proposition) that the second-order relation changes (if the second-order relation is a *power*) or leaves intact (if the relation is a *disability*).[137] As described earlier, a first-order legal proposition contains two legal actors, a specific first-order Hohfeldian legal relation (e.g., *strict-right*, *no-right*), and a state of affairs.[138] Second, there is a legal actor who holds a *power* (or not) and a legal actor who is subject to a *power* (or not).[139] Often, though not always, the two actors that are subject to the second-order relation (hereinafter, "second-order legal actors") are the same two actors subject to the first-order relation.[140] (For simplicity, I assume this is the case herein.[141]) Third, there is a specific second-order

[132] *See* Maretk Sergot, *Normative Positions* 47–48, IMPERIAL COLL. OF SCI., TECH, AND MED., 1998) (Working Paper), https://www.doc.ic.ac.uk/~mjs/publications/NormPos_Handbook.pdf (comparing the Hohfeldian notion of power with those of modern formal logical theories).

[133] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 52 (explaining the concept of Hohfeldian powers in the context of property transfer).

[134] *See id.* (explaining the timing of legal transfers).

[135] *See id.* (describing the change of legal relations following the exercise of a Hohfeldian power).

[136] *See* LINDAHL, *supra* note 12, at 203 (describing Kanger's "Power(p, q, F)," which implicitly contains four elements).

[137] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14 at 61 (describing how a "legal power usually changes . . . the 'first-order' legal relations (rights, privileges, duties, and no-rights)").

[138] *See supra* note 120 and accompanying text (explaining the tripartite structure of first-order legal relations).

[139] Like the first-order relations, I assume a second-order relations concerns only two legal actors. *See supra* note 112 and accompanying text.

[140] One notable exception are powers held by the State to change relations among private actors. For instance, if legislature passes a law that makes the sale of goods under an otherwise valid contract illegal, then the legal obligations become null and void. *See generally* David Adam Friedman, *Bringing Order to Contracts Against Public Policy*, 39 FLA. ST. U. L. REV. 563 (2012) (discussing the policy, doctrine, and cases relating to contracts for illegal goods and actions). The State in this instance would exercise a second-order power relative to two different parties to change those parties' first-order duties to Hohfeldian privileges. *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 21 n.14 (describing the State's ability to exercise legal powers to change first-order legal relations).

[141] I also assume that the actor who holds the *power* (or *disability*) is the same actor who holds the *strict-right* (or *no-right*). Of course, the actor who holds the *duty* (or *privilege*) may hold a

legal relation (e.g., *power*, *disability*) that concerns the two second-order legal actors.[142] Finally, there is a state of affairs, the occurrence of which allows the second-order relation to have effect (in the case of a *power*) or no effect (in the case of a *disability*) on the first-order legal relation.[143] This "second-order" state of affairs, like the first-order state of affairs, is a specific factual state of the world, at some instant t or over period of time Δt (or multiple time periods), which is typically a specific action undertaken by the legal actor holding the *power* (or not).[144]

The general form of any second-order legal proposition is as follows:[145]

[Actor #1][second-order legal relation][Actor #2] (a state of affairs that effects A's power (or not)) [a first-order legal proposition] (8)

Using the notation from the previous section:

$$J_2 = (X_{j2}Y(S_2))(J_1) \text{ where it is understood that } J_1 = X_{j1}Y(S_1) \quad (9)$$

As in the case of the first-order relations, in view of the "correlativity" principle of the Hohfeldian framework, by convention, it is possible to orient the form for J_2 always in "*power* notation," meaning that X is the

power (or *disability*) to change the relevant first-order relation. It is straightforward to modify the formalism presented herein to take account of this possibility.

[142] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 52 (discussing the exercise of a Hohfeldian power as between two legal actors X and Y).

[143] *See id.* at 47–48 (setting forth and discussing Hohfeld's position that a certain set facts termed "operative facts" will effectuate the exercise of a power).

[144] Generally, if there is *any* state of affairs *whatsoever* that exists that provides A with *a power* to change a given first-order legal relation, A is said to have a "power." In the above description, a complete second order relation concerns whether the occurrence of a *specific* state of affairs provides A an ability (or not) to exercise its *power*. Thus, even if A is said generally to have the "power" to effectuate a transfer of L, some states of affairs (e.g., A attempts to transfer L to B by an oral contract) may not specifically implement a *power*. An elementary second-order legal proposition concerns a *specific* state of affairs, and *not* whether there is at least one state of affairs that provides A with a *power* over the relevant first-order relation. However, it is straightforward to generalize the formalization proposed herein to take into account multiple states of affairs.

[145] Importantly, a second-order legal proposition must always concern a first-order legal proposition. As Hohfeld noted, in some instances, operative facts not under the volitional control of legal actors may result in a change of a lower-order legal relation. *See id.* (discussing the role of volitional control in the exercise of a legal power). In such cases, one may also formulate second-order legal propositions that simply depend on whether a general state of affairs obtains (or not), eliminating the legal actors who exercise and are subject to Hohfeldian liabilities, respectively. I ignore such "internally" driven changes in legal relations in what follows for simplicity.

legal actor that holds a *power* (or not) and Y is the legal actor that is subject to a *liability* (or not).[146] In this event:

$$j_2 = p \text{ or } {\sim}p \tag{10}$$

Thus, in this *power* notation, any second-order relation must take the form:[147]

$$J_2 = (X\ {}_{j_2}Y(S_2))(J_1) \text{ where } j_2 = p \text{ or } {\sim}p \tag{11}$$

The application of any precise legal rule concerning a second-order relation involving two legal actors and a specific state of affairs, S_2, may be expressed in the form of (11).[148]

In certain instances, an actor may have a third-order *power* to change a second-order *power* relation.[149] For example, a state regulatory agency may exercise a third-order *power* to annul a landowner's second-order *power* to transfer land in certain circumstances (e.g., in the event the land is environmentally contaminated).[150] In such case, the regulatory agency's exercise of this third-order *power* would change the landowner's second-order *power* to a *disability* (i.e., from p to ${\sim}p$).[151]

In general, second or higher-order legal proposition may be expressed in *power*-notation as follows:[152]

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[146] *See supra* note 121 and accompanying text (adopting a "rights" notation for first-order relations wherein the rights-holder appears first in the legal proposition).

[147] Recall the assumption that the power holder and liability holder are the same legal actors as those appearing in the applicable first-order legal relation, J_1, subject to the power (or not). More generally, any third-party may hold a power (or not) with respect to any underlying first-order legal relation. In this case, the legal relation J_2, would refer solely to one power-holder, e.g., W, which in turn would holder power (or not) with respect to the relation J_1. By implication, both parties subject to the relation, J_1, would have liabilities (or immunities) with respect to W's power (or disability).

[148] *Cf. supra* notes 122–123 and accompanying text (explaining how first order relations concern two legal actors, a first-order legal relation, and a specific state of affairs).

[149] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 61 n.61 (briefly discussing higher-order powers).

[150] *See, e.g.*, Superior Air Prod. Co. v. NL Indus., Inc., 522 A.2d 1025, 1026 (N.J. Super. Ct. App. Div. 1987) (noting that title to the land at-issue could not be transferred under New Jersey's Environmental Cleanup Responsibility Act (ECRA) until the Department of Environmental Protection (DEP) certified that the owner had "rectif[ied] the contamination" present on the land).

[151] Alternatively, as in *Superior Air Products Co. v. NL Industries, Inc.*, an owner might be disabled from selling land until the government affirmatively authorizes the sale, converting the disability to a power. *See id.*

[152] In other words, third-order and even higher-order powers operate on lower-order powers (or disabilities) to change those powers to disabilities (and vice-versa). For instance, a fourth-order constitutional amendment power might change a third-order constitutional power that allows a legislature to abridge a private party's second-order power to transfer land. A fifth-order

$$J_n = (X_{j_n} Y(S_n))(J_{n-1}) \text{ where } j_n = p \text{ or } {\sim}p \text{ and } n > 1 \qquad (12)$$

Importantly, all *n*th-order *powers* (or *disabilities*) affect (or not) an *(n-1)*th-order legal relation.[153]

III. A MATHEMATICAL FORMALIZATION OF THE HOHFELDIAN FRAMEWORK

The description so far is essentially the same as the deontic logic formalisms of Hohfeld, suitably supplemented with a logic of legal powers.[154] Unlike the logical formalizations of the Hohfeldian system, this section proposes a novel, *mathematical* formalization.[155] Such an approach naturally lends itself to quantitative measures, such as entropy and temperature, and can readily incorporate the language of quantum mechanics to model legal indeterminacy.[156]

A. Classical Mathematics of the Hohfeldian Relations

The first step is to link the first-order relations (e.g., *strict-rights*) to the higher-order relations (e.g., *powers*).[157] Noticing that the notation in (12) for *powers* is similar in form to (6) for *strict-rights*, by designating a first-order *strict-right* as r_1, and a second-order *power* as r_2, and so forth, then any *n*th-order legal proposition, J_n, may be expressed as follows:

amendment power might effectuate changes in the fourth-order amendment power. *See generally* F.E. Guerra-Pujol, *Gödel's Loophole*, 41 CAP. U. L. REV. 637 (2013) (addressing the logician's Kurt Godel's assertion that the amendment power of the U.S. Constitution may be subject to self-amendment).

153 *See generally* Visa A.J. Kurki, Comment, *Hohfeldian Infinities: Why Not to Worry*, 23 RES PUBLICA 137 (2017) (examining the orders of Hohfeldian relations and addressing the potential problem of "infinite regress" of these orders).

154 *See supra* Part II.B; *see also supra* notes 145–52 and accompanying text (briefly describing a logic of Hohfeldian powers).

155 *See supra* notes 5, 40 (discussing the prior literature).

156 *See infra* Parts IV–V.

157 Proposition (13) relates powers to rights because, by definition, a "strict-right" notation was chosen for the first-order relations wherein X is the right-holder (or not) and Y is the duty-holder (or not). However, a "duty" notation could have been chosen for the first-order relations, resulting in a power-duty relation in (13). As the remainder of this Part shows, the mathematics of a power-duty approach would be exactly the same as a power-right approach, making the two approaches equivalent. Nonetheless, from an interpretive perspective, because "power" and "right" are similar with respect to the "control" of a legal actor, it is arguably more sensible to adopt the power-right notation. *See infra Part III*; *see also* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 55 ("a power bears the same general contrast to an immunity that a right does to a privilege").

$$J_n = (X_{j_n} Y(S_n))(J_{n-1}) \text{ where } j_n = r_n \text{ or } {\sim}r_n,\ n > 0, \text{ and } J_0 = 1 \quad (13)$$[158]

Notably, previous formalisms have required *both* the *strict-right* and *power* relations as atomic components of the Hohfeldian system.[159] In the formalism of (13), *strict-rights* are species of *power*—or, vice-versa, depending on one's perspective.[160] Thus, the Hohfeldian formalism only requires one atomic legal relation (e.g., *power*), plus the notions of negation, correlativity, and order to generate the remaining legal relations.[161] Furthermore, using the formalism of (13), the legal relations j_n may be formalized mathematically as tensors—mathematical objects commonly employed in general relativity, quantum mechanics, and information theory.[162] In this section, I describe the basic tensor properties of the classical legal relations.[163]

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[158] That $J_0 = 1$, rather than 0, means that legal proposition is active, or in other words, "exists." Thus, J_0 can be viewed as an "existence bit." Although an ordinary *power* will only *change* an already-existing legal relation (i.e., leave its existence intact), a termination power may *terminate* a lower-order relation (i.e., change J_0 from 1 to 0) and a creation power will *create* a lower-order relation (i.e., change J_0 from 0 to 1). The creation and termination of legal relations will typically occur only upon the creation or dissolution of a specific legal actor (e.g., a person is born, resulting immediately in a host of new legal relations to others), or the creation or elimination of the existence of a specific state of affairs (e.g., a building is erected on a piece of land, resulting in *duties* of non-owners not to damage it). *See, e.g.*, Mary Patricia Byrn & Jenni Vainik Ives, *Which Came First the Parent or the Child?*, 62 RUTGERS L. REV. 305, 307 (2010) ("the moment [a child] is born, she is a legal person endowed with constitutional rights"). One may compare these special power operators to the creation and annihilation operators of quantum field theory, which create or annihilate quantum states. *See* GORDON BAYM, LECTURES ON QUANTUM MECHANICS 411–17 (2018) (describing creation and annihilation operators). Notably, *changing* a *right* into a *no-right* relation is *not* a termination of a relation; conversely, changing a *no-right* to a *right* is *not* a creation of a relation. *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 47 n.53 (discussing the difference between the change, creation, and termination of legal relations). Rather, those operations are ordinary changes in legal relations. The remainder of the discussion refers only to powers that change legal relations.

[159] *See* Andrew Halpin, *Hohfeld's Conceptions: From Eight to Two*, 44 CAMBRIDGE L.J. 435, 435 (1985) (positing that Hohfeld's eight legal relations are reducible to two via the logical relations that inhere among them).

[160] Thus, Hohfeld's reflection that "a power bears the same general contrast to an immunity that a right does to a privilege" is not merely one that is based in conceptual similarity via the notion of "control," but is grounded formally via a logical relation that connects the orders of legal relations. *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 55.

[161] As Halpin notes, negation and correlativity are sufficient to reduce each order of Hohfeldian relations from four to one, leaving two relations, such as power and right. *See* Halpin, *supra* note 159, at 456–57. A formal treatment of order reduces the remaining two to one.

[162] *See generally* ROBERT C. WREDE, INTRODUCTION TO VECTOR AND TENSOR ANALYSIS (2013) (describing tensors and their applications).

[163] For a useful treatment of the more general relationship between logic and matrix algebra, including tensors, *see* STERN, *supra* note 20 (formulating propositional and other logics using the mathematical formalism of matrix algebra).

i. Formalizing the First-Order Relations

Recall for a first-order proposition, J_1, that in *strict-right* notation:[164]

$$J_1 = (X_{j_1} Y(S_1))(J_0) \text{ where } j_1 = r_1 \text{ or } {\sim}r_1 \text{ and } J_0 = 1 \tag{14}$$

Because the classical, specific first-order relation j_1 can only be one of two values (either a *strict-right* (r_1) or a *no-right* ($\sim r_1$)), one can represent j_1 by a simple binary object, such as a classical bit of information.[165] For instance, one can represent a *strict-right* as an "on-bit" (in binary notation, the number "1") and a *no-right* as an "off-bit" (in binary notation, the number "0").[166] In order to more easily manipulate these bits mathematically, it is useful to adopt an equivalent vector formalism,[167] wherein:

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } {\sim}r_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ (matrix notation)} \tag{15}$$

In the more familiar Cartesian coordinate notation, $r_1 = (1, 0)$ and $\sim r_1 = (0, 1)$.[168] (See Fig. 9.)

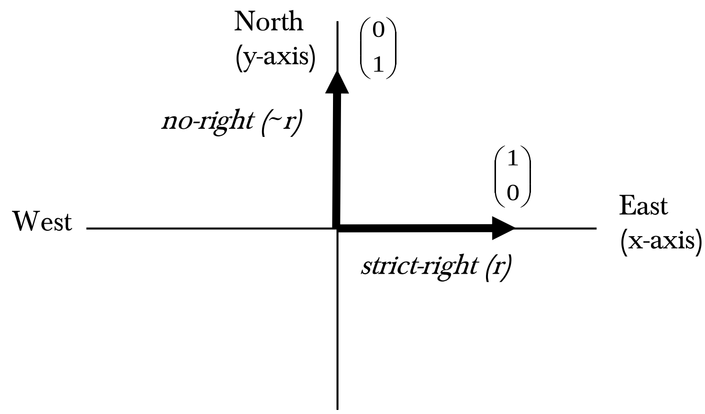
[164] *See supra* notes 122, 152 and accompanying text (defining first-order zeroth-order legal propositions).

[165] *See* DAN C. MARINESCU & GABRIELA M. MARINESCU, CLASSICAL AND QUANTUM INFORMATION 221–344 (2011) (discussing classical information theory in the context of quantum information theory).

[166] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 7–8 (aligning the classical Hohfeldian typology with classical information bits). Note that although the level of damages that may be awarded in a judgment is continuous, liability—which is binary—is a precondition to the award of damages. *See generally* Stephen A. Smith, *Duties, Liabilities, and Damages*, 125 HARV. L. REV. 1727, 1728–30 (2012) (exploring duty- and liability-approaches to the payment of damages).

[167] *See* JONATHAN A. JONES & DIETER JAKSCH, QUANTUM INFORMATION, COMPUTATION AND COMMUNICATION 7 (2012) (describing a classical bit as a point on the north pole or the south pole on the Bloch sphere). Instead of using the Bloch sphere approach, here I align the classical bits with the vector notation of quantum spin around the z-axis, as it leads to a more direct alignment with the Pauli matrix approach of quantum mechanics. *See infra* notes 211–216 and accompanying text. Specifically, because the Bloch sphere is a 2-dimensional surface, one can use an equivalent description with two-dimensional vectors in the base {|1>,|0>}. Therefore |1> (1,0) and |0> (0,1). *See infra* notes 211–16.

[168] *See* RUTHERFORD ARIS, VECTORS, TENSORS, AND THE BASIC EQUATIONS OF FLUID MECHANICS 8–15 (Dover Publ'ns, Inc. 2012) (1962) (describing Cartesian vectors).



As shown in this figure, a vector is an object with a specific length and direction. Here, the *strict-right* and *no-right* vectors are (1,0) and (0,1), respectively. Each has a length of one unit and points in an easterly and northerly direction, respectively.

Fig. 9. A Two-Dimensional Classical Representation of Strict- and No-Right Vectors

In more general mathematical terms, vectors are first-rank tensors.[169] Thus, a specific first-order relation (i.e., *strict-right* or *no-right*) can be represented formally by a first-rank tensor.[170]

ii. Formalizing the Classical Second-Order Relations

Recall that a second-order legal proposition written in *power* notation must either contain a *power* (which changes a first-order relation) or a *disability* (which leaves intact a first-order relation):[171]

[169] *See* A.I. BORISENKO & I.E. TARAPOV, VECTOR AND TENSOR ANALYSIS WITH APPLICATIONS 61–63 (Richard A. Silverman ed. & trans., Dover Publ'ns, Inc. 1979) (1968) (describing vectors as "first-order" tensors). Interestingly, zeroth-order tensors are scalars, i.e., single numbers, which provides a mathematical mapping between the first-order legal relations and the existence of those relations (or not) via the existence bit for legal relations, J_0. *See supra* note 158 and accompanying text (positing the legal existence bit).

[170] *See supra* note 167 and accompanying text. Although representing classical legal relations with vectors and tensors is not essential, it provides a useful explanatory linkage between the classical and quantum legal relations, which in turn is useful to better understand the nature of the physical laws. *See infra* Parts III.B, VI.B.

[171] *See supra* notes 145–146 and accompanying text (defining a second-order legal proposition).

$$J_2 = (X_{j2}Y(S_2))(J_1) \text{ where } j_2 = r_2 \text{ or } {\sim}r_2 \tag{16}$$

In the event that $j_2 = r_2$ (i.e., j_2 is a *power*), then *X*'s exercise of the *power* will transform the specific legal relation of J_1 (i.e., j_1) into its negation.[172] So if j_1 is a *strict-right* (i.e., the vector (1,0)), the operation of a *power* on j_1 will transform the *strict-right* into a *no-right* (i.e., the vector (0,1)).[173] Thus, j_2 can be thought of as a mathematical operator that either transforms a first-order vector relation into its negation (i.e., in the event j_2 is a *power* (r_2)) or leaves a first-order vector relation intact (i.e., in the event j_2 is a *disability* (~r_2)).[174]

In this regard, one can represent a second-order *power* (r_2) by a second-rank tensor that transforms

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ into } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and vice-versa, and a second-order *disability* (~r_2) by a second-rank tensor that leaves the first-order relations (i.e., vectors) intact.[175] In matrix notation, r_2 and ~r_2 are represented by the following forms:

$$r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } {\sim}r_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

In other words, in matrix notation, r_2 (a second-order *power*) is a 2 x 2 permutation matrix (i.e., it flips the first and second components of the vector it multiplies), and ~r_2 (a second-order *disability*) is a 2 x 2 identity

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172 *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 43 (explaining how legal powers "change . . . a given legal relation").

173 *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 52 (noting how in the context of the transfer of personal property, a power is effectuated that changes a privilege to use the property to an obligation not to use it).

174 The mathematical operators that transform a *strict-right* into a *no-right* (and vice-versa) are mathematical functions that operate on vectors as inputs, whereby the components of the input vectors are transformed linearly by the operators into outputs. *See* MICHAEL A. NIELSEN & ISAAC L. CHUANG, QUANTUM COMPUTATION AND QUANTUM INFORMATION 62–64 (10th ed. 2010) (describing linear operators in the context of quantum mechanics).

175 These are two of the Pauli matrices, which appear frequently in quantum mechanics. Specifically, r_2 is equivalent to σ_1, and ~r_2 is equivalent to σ_0. These matrices can be used to construct more general matrices to allow for the transformation of various quantum states. Because the only transformations needed in a classical system are "bit flips," i.e., 0 to 1 and 1 to 0, the only necessary Pauli matrices are σ_0 and σ_1. *See id.* at 64, 460–61 (describing the Pauli matrices in their using in transforming quantum states).

matrix (i.e., it leaves intact the components of the vector it multiplies).[176] Specifically:

$$(r2)(r1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = {\sim}r1 \tag{18a}$$

$$({\sim}r2)(r1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = r1 \tag{18b}$$

$$(r2)({\sim}r1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = r1 \tag{19a}$$

$$({\sim}r2)({\sim}r1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = {\sim}r1 \tag{19b}$$

These mathematical relationships precisely reflect the exercise of a Hohfeldian power (or not). Specifically, exercising a power with respect to a right will change the right to a no-right.[177] Exercising a power on a no-right changes it to a right.[178] An attempt to exercise a power but not actually having that power (i.e., being "disabled" in the Hohfeldian sense from exercising a power) does not change an underlying legal relation.[179]

[176] *See* ROGER A. HORN & CHARLES R. JOHNSON, MATRIX ANALYSIS 6, 32 (2d ed. 2012) (explaining identity and permutation matrices). These matrix operations are also reflected in the concept of logic gates, which are abstract representations of actual, electronic gates that operate on underlying data components in a computer. *See* CIARAN HUGHES ET AL., QUANTUM COMPUTING FOR THE QUANTUM CURIOUS 49–51 (2021) (describing classical and quantum logic gates). In this sense, the operations of second-order legal powers more generally can be represented as a set of logic gates operating on underlying, lower-order data arrays (i.e., vectors) representing first-order legal relation states. *See* Stern, *supra* note 20, at 67–69 (describing the use of matrix arrays representing logic gates to transform truth-vectors from one state to another). Higher-order operations of law, *see infra* notes 180–184 and accompanying text, can be represented by a series of higher-order gates operating on lower-order gates. Whether in classical or quantum computation—and, hence, whether in classical or quantum approaches to legal relations—these gates can be constructed from linear combinations of the Pauli matrices (assuming the identity matrix is included). *See, e.g.*, J.A. Jones, Quantum Information 17–21 (2010) (unpublished research paper) (on file with the University of Oxford Department of Physics).

[177] *See supra* equation (18a).

[178] *See supra* equation (19a).

[179] *See supra* equation (18b).

iii. Higher-Order Relations

Higher-order relations (i.e., n > 2 for j_n) follow the same principles. For instance, a third-order *power*, *r3*, will transform *r2* (a second-order *power*) into *~r2* (a second-order *disability*), and vice-versa, and a third-order *disability* (*~r3*) will leave second-order relations intact.[180] Thus:[181]

$$(r3)(r2) = \sim r2 \text{ and } (r3)(\sim r2) = r2 \qquad (20)$$

$$(\sim r3)(r2) = r2 \text{ and } (\sim r3)(\sim r2) = \sim r2 \qquad (21)$$

The most general mathematical entities that correspond to *r3* and *~r3* are fourth-rank tensors, specifically a fourth-rank "Hohfeldian" permutation tensor and a fourth-rank identity tensor, respectively.[182] Unlike a two-dimensional matrix, a fourth-rank tensor (in two dimensions) will have sixteen components and is represented by a 2 x 2 x 2 x 2 four-dimensional block.[183] Nonetheless, multiplication can be carried out according to well-known rules.[184] Using these rules, *r3* acts as a fourth-rank Hohfeldian permutation tensor (i.e., a tensor that specifically flips 1's to 0's and vice-versa for a second-order relation), and *~r3* acts as a fourth-rank identity tensor (i.e., a tensor that leaves 1's and 0's intact).[185] Thus:

$$(r3)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (22)$$

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180 *See supra* notes 149–151 and accompanying text (discussing third-order powers).

181 In actuality, equations of transformation for *r2* via *r3* deserve a more complex notation. *See, e.g.*, BORISENKO & TARAPOV, *supra* note 169, at 105 (describing tensor contraction for third-rank tensors). I have simplified the description in the text for ease of exposition.

182 Also note that a Hohfeldian permutation tensor is different from the traditional Levi-Civita permutation tensor, because a Hohfeldian permutation tensor merely permutes the elements of the lower-rank tensors but cannot change the sign of any components of those tensors. *See, e.g.*, BORIS KOSYAKOV, INTRODUCTION TO THE CLASSICAL THEORY OF PARTICLES AND FIELDS 25 (2007) (describing fourth-rank Levi-Civita tensors).

183 *See* SEIICHI NOMURA, MICROMECHANICS WITH MATHEMATICA 17–18 (2016) (noting a fourth-rank, two-dimensional tensor has sixteen components). For a worthwhile conceptual exploration of the fourth and higher dimensions, *see* RUDY RUCKER, THE FOURTH DIMENSION: A GUIDED TOUR OF THE HIGHER UNIVERSES (1984).

184 *See* NOMURA, *supra* note 183, at 17–18 (multiplying tensors to effectuate a coordinate transformation).

185 *See, e.g.*, WEIMIN HAN & B. DAYA REDDY, PLASTICITY: MATHEMATICAL THEORY AND NUMERICAL ANALYSIS 11 (2d ed. 2013) ("A fourth-order tensor *C* may be defined as a linear operator mapping the space of second-order tensors into itself."); JUAN RAMÓN RUÍZ-TOLOSA & ENRIQUE CASTILLO, FROM VECTORS TO TENSORS 78–79 (2005) (discussing fourth-order tensors); WOLÉ SOBOYEJO, MECHANICAL PROPERTIES OF ENGINEERED MATERIALS 106–07 (2002) (explaining that "if a second-order tensor . . . is a linear function of another second-order function, they are related by a fourth tensor").

$$(\sim r3)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{23}$$

Given the particular properties of the classical Hohfeldian legal relations, however, one may use a much simpler approach to describe higher-order relations than higher-rank tensors. Specifically, the third-order relations (*r3* and *~r3*), as well as any higher-order powers and disabilities, can simply be written in the same format as the second-order power and disability matrices. Thus, an *n*th-order classical power (i.e., for n > 1) will always be a second-rank Hohfeldian permutation tensor and an *n*-th rank classical disability will always be a second-rank identity tensor.[186]

The tensor formulation of legal relations and propositions is not merely of theoretical interest.[187] Rather, using this formalization, many mathematical and physical variables regularly used to describe the properties of physical systems can be readily adapted to describe individual legal propositions and legal systems.[188] Such properties include entropy, indeterminacy, temperature, and information content.[189] Before I consider these properties, I describe a Hohfeldian mathematical system that incorporates legal indeterminacy using the language of quantum mechanics.

B. A Mathematical Formalization of the "Quantum" Hohfeldian Framework

i. Quantum Spin as an Analog to Post-Classical Legal Relations

As noted earlier, the classical Hohfeldian relations are binary in the sense that a legal actor holds a *strict-right* or not, a *power* or not, and so forth.[190] Even under a purely classical approach, from at least a practical perspective, legal observers often cannot perfectly predict how a given set of laws might apply to a given set of facts.[191]

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[186] For the first-order relations (i.e., $n = 1$), the mathematical representations are the first-rank tensors (i.e., vectors) described earlier. A zeroth-order relation is the scalar, J_0, or the existence bit, which represents whether a given relation exists or not, and may be altered by suitable creation and termination operators. *See supra* note 158.

[187] *See infra* Part IV (describing practical applications of the mathematical formalism introduced here).

[188] *See infra* Part IV (adapting physical properties to describe the properties of legal systems).

[189] *See infra* Part IV.

[190] *See supra* Part III.A (discussing classical, binary first-order and higher-order legal relations).

[191] *See supra* notes 29–33 and accompanying text (discussing legal indeterminacy).

Such uncertainty in outcomes can be modeled via the use of classical notions of entropy, and the Hohfeldian relations may correspondingly be expressed in terms of classical probabilities.[192] Importantly, on a purely classical approach, although legal relations may be uncertain in practice, they are not in principle.[193] In other words, on such a purely formalist view, legal relations are like the proverbial blue and red marble held by a child in tight-fisted left and right hands, respectively.[194] Although the observer does not know which hand contains which marble, there is a determinate "correct" answer to the question of whether the child's left or right hand contains the red marble, just as there is a determinate, "correct" answer to how a given set of laws applies to a given set of facts.[195]

On a post-classical, legal realist view of legal relations, the outcome of a legal dispute is often indeterminate not only in practice, *but also in principle*, prior to a final adjudication by the judicial system.[196] Following this approach, such an indeterminate legal relation becomes a probability distribution of possible results and is only resolved into a determinate relation only upon final judgment.[197]

The probability distribution in this case does not merely result from a lack of knowledge of an underlying, "true" legal relation.[198] The implication is that legal relations—at least a good share of them—are fundamentally indeterminate until a final decision is issued by the court,

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[192] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 7–8 (examining classical versus quantum approaches for probabilistic Hohfeldian relations).

[193] Like statistical mechanics, which applies classical mechanics to multi-particle systems in which macroscopic properties of the collection of particles are known, but the microscopic properties of individuals particles are not—merely due to practical difficulties—in classical legal approaches, indeterminacy is purely epistemic in nature. *See generally* Y.M. GUTTMANN, THE CONCEPT OF PROBABILITY IN STATISTICAL PHYSICS 94 (1999) (characterizing classical physical systems subject to probabilistic laws as "stochastic" rather than "indeterminate" in nature).

[194] *See generally* ARIEH BEN-NAIM, DISCOVERY ENTROPY AND THE SECOND LAW OF THERMODYNAMICS: A PLAYFUL WAY OF DISCOVERING A LAW OF NATURE 73–76 (2010) (describing probability in thermodynamics using a red marble and blue marble randomly thrown into cells).

[195] *See* PATRICK A. HEELAN, QUANTUM MECHANICS AND OBJECTIVITY: A STUDY OF THE PHYSICAL PHILOSOPHY OF WERNER HEISENBERG 75 (2012) (noting that in statistical mechanics, "the state of the system is . . . determinate but still unknown").

[196] *See supra* notes 31–33 and accompanying text; *see also* Max Radin, *Permanent Problems of the Law*, 15 CORNELL L.Q. 1, 17 (1929) (noting that to reliably predict judgments, one would need to assume that "all judges are alike, that every judge acts uniformly, [and] that new situations as they arise are exactly like, or almost exactly like, old situations already judged").

[197] *See infra* notes 245–254 and accompanying text (proposing a formal model of probabilistic legal relations and associated judgments).

[198] *See generally* KENT GREENAWALT, LEGAL INTERPRETATION: PERSPECTIVES FROM OTHER DISCIPLINES AND PRIVATE TEXTS 42 (2010) ("Epistemic indeterminacy may exist if highly reasonable people, as well informed as is practical, have an unresolvable disagreement about whether [something] is correct, or have no idea whether it is correct.").

and are unknowable until that time.[199] Such inherent, ontological indeterminacy can arise from "vagueness or indeterminacy of legal doctrine," "uncertainty as to the impact evidence will have on the decisionmaker," idiosyncratic behavior in enforcement and adjudication, and the influence of unknowable, extra-legal factors on the regulatory and judicial process.[200]

Like the debates regarding physical indeterminacy, there is no clear answer to whether legal indeterminacy is ontological or merely epistemic in nature.[201] Yet, whatever one's view, from a mathematical perspective, one can always posit the most general framework as a starting point and simplify afterwards to the extent doing so is justified and useful.[202] In this situation, the quantum mechanical framework used to describe the inherent indeterminacy of physical systems is more general than the classical framework used to describe merely stochastic behavior.[203] More specifically, the quantum framework in the "macroscopic" limit can effectively reproduce classical behavior, but not vice-versa.[204] In addition, one primary goal of this Article is to use the formalism of social law proposed here to better understand the structure of physical law.[205] Thus, whatever one's take is on the "true" nature of legal adjudication, the

[199] *See* Kennedy, *supra* note 31, at 385 ("The judge cannot claim that legislative acquiescence legitimizes his action because he himself creates, through his decision of particular cases, the situation from which will emerge an as yet indeterminate constellation of legislative power.").

[200] *See* Charles M. Yablon, *The Good, the Bad, and the Frivolous Case: An Essay on Probability and Rule 11*, 44 UCLA L. REV. 65, 93 n.71 (1996).

[201] *See supra* notes 31–33 and accompanying text (citing references espousing various views on legal indeterminacy).

[202] *See generally* Robert L. Causey, *Professor Bohm's View of the Structure and Development of Theories*, *in* THE STRUCTURE OF SCIENTIFIC THEORIES 398 (Federick Suppe ed., 2d ed. 1977) ("With suitable auxiliary hypotheses, a good, general theory can have many successful applications.").

[203] *See* Abdo Abou Jaoudé, *The Paradigm of Complex Probability and Isaac Newton's Classical Mechanics: On the Foundation of Statistical Physics*, *in* THE MONTE CARLO METHODS: RECENT ADVANCES, NEW PERSPECTIVES AND APPLICATIONS 48 (Abdo Abou Jaoudé, ed., 2022) ("[C]lassical mechanics [is] an approximate theory to quantum mechanics which is a more general theory.").

[204] PAVEL BÓNA, CLASSICAL SYSTEMS IN QUANTUM MECHANICS vi (2020) (noting the book describes how to "deriv[e] a classical (macroscopic) time evolution (which is, in general, in a certain sense stochastic . . .) from the underlying reversible quantal dynamics").

[205] *See infra* Part VI (using the mathematical formalism introduced herein for social law to describe the structure of scientific law).

quantum framework introduced here proves useful for this broader aim.[206] As such, I adopt the post-classical approach to legal relations.[207]

Importantly, as others have recognized (but have not formalized), this post-classical description of legal relations is arguably analogous to quantum mechanical formulations of the properties of physical objects.[208] In particular, the mathematical description of the "quantum spin" of an electron in one dimension is particularly well-suited for describing post-classical legal relations. Although quantum spin is an inherent property of particles with no classical analog, in effect, spin is measured in discrete units along a given-axis in either a "clockwise" or "counterclockwise" fashion, which in turn can be used to formally model the post-classical, probabilistic Hohfeldian first-order relations.[209] If an electron is measured to be spinning clockwise, then by definition it has a "down" spin, whereas if it is measured to be spinning counterclockwise, then it has an "up" spin.[210] Like first-order legal relations, up and down spins (here, around the z-axis) of an electron can be represented by vectors:[211]

$$\psi_{z+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \psi_{z-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{24}$$

In (24), ψ_{z+} represents a spin-up state and ψ_{z-} represents a spin-down state.[212] These vectors can be defined to correspond to the classical legal

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[206] *See infra* Part VI.B (applying the quantum framework to posit "second-order" physical processes that may explain quantum measurement).

[207] In the event that somehow down the line legal observers conclusively show that legal indeterminacy is merely stochastic—which seems highly unlikely—then the only change in the discussion that follows would be to interpret the probabilistic aspects as merely reflecting uncertain knowledge, rather than inherent uncertainty. Although there are "quantum-like" forms of interference and entanglement present in legal systems, if they are not fundamentally quantum in the sense of physical law, they may be described by a classical probabilistic formalism. *See* MARTIN CONCOYLE & G.P. COATMUNDI, THE MATHEMATICAL STRUCTURE OF STABLE PHYSICAL SYSTEMS 110 (2014) ("It is well known that the math techniques of classical physics cannot be used to describe quantum systems.").

[208] *See* Joseph Blocher, *Schrödinger's Cross: The Quantum Mechanics of the Establishment Clause*, 96 VA. L. REV. 51, 55 (2010) ("What reality exists before judges render judgment?"); *see also* Atik & Jeutner, *supra* note 40, at 13–14; *see generally* Felix S. Cohen, *Transcendental Nonsense and the Functional Approach*, 35 COLUM. L. REV. 809, 827 (1935) (suggesting a "parallel between the functional method of modern physics and the program of realistic jurisprudence").

[209] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 7–8 (briefly introducing the formalism presented here).

[210] WALTER GREINER, QUANTUM MECHANICS: AN INTRODUCTION 333–36 (2011) (providing an introductory account of quantum spin).

[211] *See id.* (describing the z-axis spin states).

[212] *See id.* (describing up and down spin states).

relations *strict-right* and *no-right*, respectively.[213] In what follows, a spin-up state corresponds to a *strict-right* and spin-down state to a *no-right*.[214] Like a ontologically probabilistic legal relation before a court pronounces *strict-right* or *no-right* in a final judgment, on the traditional view of quantum mechanics, quantum spin is a ontologically probabilistic *superposition* of both the up and the down spin states until a so-called "measurement" is performed.[215] Just like a final judgment results in *either* a *strict-right or* a *no-right* outcome, a measurement of an electron's spin along the z-axis results *either* in an *up or* a *down* state, "collapsing" the existing probability distribution into a single, definite state.[216]

Importantly, if we adhere to the fully post-classical nature of legal relations and the standard interpretation of quantum mechanics, both legal relations and electron spin states are unknowable in principle prior to a

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[213] *See generally* STERN, *supra* note 20, at 152–54 (describing "spinor logic" by associating the state of a spin vector with binary logic vectors, which allow the association of spin states with true-false binary values).

[214] *See id.* (noting that the Pauli spinor matrices may act on the spin states in a manner akin to the logical operators in matrix notation, which supports the proposal here that Hohfeldian relations may be described by spin states and the associated Pauli matrices).

[215] *See* SHABNAM SIDDIQUI, QUANTUM MECHANICS: A SIMPLIFIED APPROACH 48 (2018) (describing the "collapse" of superpositions of spin states upon measurement to a single value).

[216] *See id.* The term "superposition" is used in a variety of senses in physics and mathematics. A very narrow usage concerns the scalar and additive nature of potential solutions to linear equations that govern physical particles, fields, and systems. *See, e.g.*, STANLEY J. FARLOW, AN INTRODUCTION TO DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS 142 (Dover Publ'ns, Inc. 2012) (1994). A usage popular in quantum mechanics extends beyond the mathematical definition by referring to a "superposition" as a "pure" quantum state that has relative complex phases of possible "eigenstates" that effectively interfere with one another as waves as the state evolves, in contrast to a "mixed" state, "which is insensitive against phase differences of its constituents." BERND THALLER, ADVANCED VISUAL QUANTUM MECHANICS 239 (2005). Even then, some physicists refer to pure states as "coherent superpositions" and mixed states as "incoherent superpositions," further complicating the matter. *Id.* Finally, "superposition" is often used to refer to particles, fields, or systems that "involve the failure of [the] system to . . . have any determinate value of . . . a single observable." Claudio Calosi & Jessica Wilson, *Quantum Metaphysical Indeterminacy*, 176 PHIL. STUD. 2599, 2603–04 (2019).

Unless indicated otherwise, this Article uses the term "superposition" in the sense of having no determinate value, irrespective of linearity and interference effects. Even still, the "failure . . . to . . . have a determinate value" may mean either epistemic indeterminacy, ontological indeterminacy, or both. *See id.* at 2603–2610. Here, "superposition" and "indeterminate" refer to ontological indeterminacy. Moreover, like a legal state that is ontologically indeterminate prior to judgment, even absent interference effects, I argue below that a quantum state may be ontologically indeterminate prior to measurement. *See infra* Part VI.B; *see generally* Lorenzo Catani, Matthew Leifer, David Schmid & Robert W. Spekkens, *Why Interference Phenomena Do Not Capture the Essence of Quantum Theory*, 7 QUANTUM 1119 (2023), https://doi.org/10.22331/q-2023-09-25-1119 (exploring the role of interference in quantum mechanics).

judgment or measurement.[217] As such, both states are not analogous, say, to a coin that has been flipped and is resting on someone's covered hand, existing either in a heads or tails state, for which an observer does not yet know the outcome.[218] Indeed, in a deterministic, classical world, even a coin that is spinning in the air will with absolute certainty land in a particular position.[219] In the classical situation, the coin is (or will be) either in a heads state or a tails state, and it is merely the observer's knowledge that is of a statistical nature.[220] For *inherently* indeterminate legal relations and quantum spin prior to judgment or measurement, there is no classical "hidden variable" that describes whether there is a strict-right or no-right or up or down spin—all that exists is a blurred superposition of probabilities.[221]

Mathematically speaking, the pre-measurement probabilistic superposition of electron spin states, analogous to the most general form of pre-adjudication legal states, can be described follows:

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[217] *See supra* notes 190–207 (discussing legal indeterminacy). The indeterminacy in law and quantum mechanics due the superposition of states that cannot be explained by some set of "hidden variables," as that term is employed here, is related to but conceptually different from the "uncertainty principle" in quantum mechanics. Specifically, ruling out "hidden variables" in this Article concerns the impossibility of knowing *prior to* judgment or measurement—even in principle—what the result of a judgment or measurement will be of any state of the system in a superposition. In contrast, the Heisenberg uncertainty principle concerns the unknowability of two non-commuting observables, such as quantum spin along two orthogonal axes, after measurement of one of the observables. *See generally* Calosi & Wilson, *supra* note 216, at 2603–04 (distinguishing between superposition, incompatible observables, and entanglement indeterminacy). Any references to "uncertainty" in this article concern "indeterminacy" of the superposition sort and not "uncertainty" in the sense of incompatible observables.

[218] *See* JEFFREY FOSS, SCIENCE AND THE WORLD: PHILOSOPHICAL APPROACHES 242 (2014) ("[A]ccording to classical physics, there is no indeterminacy . . . in the coin toss itself—the whole process is strictly limited by the laws of physics to precisely one result . . . ."). Some interpretations of quantum mechanics are epistemological in nature, contending that there are indeed hidden variables that realistically describe the quantum states of physical systems, but such theories are arguably wanting—at least if the hidden variables are first-order in nature in the Hohfeldian sense—given the variety of no-go theorems for hidden variables approaches and the plausibility of the second-order model of quantum measurement presented here. *See generally* Radin Dardashti, *No-Go Theorems: What Are They Good For?*, 86 STUD. HIST. & PHIL. SCI. PART A 47 (2021), https://doi.org/10.1016/j.shpsa.2021.01.005.

[219] *See* FOSS, *supra* note 218, at 242 (noting that the coin toss itself is deterministic).

[220] *See id.* (comparing the metaphysics of classical and quantum physics).

[221] *See generally* F.J. BELINFANTE, A SURVEY OF HIDDEN-VARIABLE THEORIES (2014) (describing a variety of hidden variable theories in quantum theory). Here, I assume that there is indeed "metaphysical" indeterminacy with respect to both legal and physical superpositions of states. Following this line of thinking, it is possible that the inherent indeterminacy in the law wholly stems from indeterminacy introduced by quantum-like effects in human decisionmaking. *See* WENDT, *supra* note 6, at ch. 8 (describing theoretical approaches that attempt to explain human decisionmaking in terms of quantum formalism). Note that such an assumption would not rule out a "second-order" hidden variables theory that deterministically explains how quantum states both evolve and are measured.

$$|\psi_z\rangle = a\,|\psi_{z+}\rangle + b\,|\psi_{z-}\rangle \text{ where } a \text{ \& } b \text{ are complex numbers} \quad (25)$$

Here, $|\psi_z\rangle$ is the overall spin (around the z-axis) of the electron, while $a\,|\psi_{z+}\rangle$ is the contribution to the overall state by up-spin and $b\,|\psi_{z-}\rangle$ is the contribution to the overall state by down-spin.[222]

The probability that upon measurement of the electron that the spin will be measured as "up" is:[223]

$$P(\psi_{z+}) = |a|^2 \quad (26)$$

Similarly, the probability of measuring the electron in a "down" state is:[224]

$$P(\psi_{z-}) = |b|^2 \quad (27)$$

Since the probability of measuring either an "up" or "down" spin is 100%, it must be the case that:[225]

$$|a|^2 + |b|^2 = 1 \quad (28)$$

One potential counterargument to this analogy is that "pure" quantum states typically self-interfere—that is, exhibit physical interference patterns like those of light even absent contact with another physical system—and that there is no analogous self-interference (or even interference with other systems) for legal relations.[226] As such, legal relations should not be constructed with complex numbers like electron spin and, instead, should simply be structured as classical probabilities, akin to "mixed" quantum states.[227]

There are two key responses. First, there is evidence that the human decisionmaking process follows a quantum-like form of cognition that exhibits interference.[228] For instance, on this view, when a jury is deciding whether a defendant is guilty, their cognitive states are in essence, or at

[222] *See* GREINER, *supra* note 210, at 333–36.
[223] *See id.*
[224] *See id.*
[225] *See id.*
[226] *See* GREGG JAEGER, QUANTUM INFORMATION: AN OVERVIEW § 1.5, at 23 (2007) (describing "the self-interference of a single-qubit system").
[227] *See id.* at 6–8 (describing mixed states).
[228] *See* Zheng Wang & Jerome R. Busemeyer, *Interference Effects of Categorization on Decision Making*, 150 COGNITION 133, 133–49 (2016) (positing that human cognition and decisionmaking adhere to quantum-like statistical rules that exhibit a form of categorization interference).

least akin to, quantum states that exhibit interference.[229] If that is the case, then legal relations—because they will depend on the decisionmaking processes of judges and juries—will also be best modeled by quantum states.[230]

Second, even if legal relations can be modeled wholly by classical statistics, there remains the ontological reality of inherently indeterminate post-classical legal relations.[231] In other words, even if there is no self- or other forms of interference involved in modeling legal relations, as the above discussion illustrates, if the post-classical view of legal relations is correct, there is no means to determine the result of judgment prior to its occurrence.[232] This is not merely an epistemological issue.[233] Rather, like the traditional interpretation of measurement of quantum spin, it is the measurement, i.e., judgment, itself that "collapses" the quantum state into one outcome or another.[234] So, at least in the sense of measurement, post-classical judgment is quantum-like in nature and the use of quantum spin to model judgment is justifiable. In sum, it is entirely possible legal states exhibit interference in the quantum sense, but even if they do not, there is good reason to model them as (one-dimensional) quantum states.[235]

The issue of measurement in quantum mechanics raises a thorny question of exactly how the process occurs. For instance, how can an electron be in a probabilistic superposition of spin states "before

[229] *See id.* at 133 ("[A] judge needs to categorize a defendant as guilty or not before assigning a punishment . . . .").

[230] *See id.* at 141–45 (finding that a quantum model best described empirical data resulting from tests of human categorization-decision tasks).

[231] *See supra* notes 196–200 (describing the post-classical view of adjudication).

[232] *See id.*

[233] *See id.*

[234] *See id.*

[235] Notably, legal relations can be modeled with only one axis of quantum spin—i.e., wholly ignoring the non-commuting observables of the other axes. If there are indeed no quantum-like interference effects for legal states, the complex coefficients of legal states simply become real coefficients that are the square root of the classical probabilities. Thus, the Born rule of squaring the coefficients to determine the probability upon measurement does not introduce any concerns in a one-dimensional spin model. In other words, in full generality, we can represent legal states with quantum spin formalism without adverse effects even in the absence of interference effects. *See generally* DAVID H. MCINTYRE, QUANTUM MECHANICS: A PARADIGMS APPROACH ch. 1 (2022) (describing quantum spin and how the Stern-Gerlach experiment illustrates inherently quantum effects when measurements across multiple spin axes are considered). Yet, such an approach does not foreclose a quantum-like aspect of measurement of post-classical legal relations—namely, the fact that judgment itself decides the measurement outcome, rather than some first-order hidden variable pre-determining the result of judgment. *See supra* notes 196–200.

measurement" and in one precise state "after measurement"?[236] No one has yet satisfactorily answered this question, and it is known as the "measurement problem" of quantum mechanics.[237] Luckily, in the world of law, there is no "measurement problem," because we know precisely how legal "measurements" are made from start to end.[238] In particular, a court will issue a final ruling, declaring once and for all (assuming there are no appeals left) the *rights* and *duties* (or lack thereof) of the parties in dispute.[239] A quantum formalization of the Hohfeldian relations follows straightforwardly from this observation.[240]

In order to anchor the formalization in a real-world example, suppose Company *A* has a patent on a cholesterol-lowering drug.[241] Company *B* is a pharmaceutical company that wishes to avoid infringing Company *A*'s patent, so it carefully reviews the patent, and attempts to "design around" it.[242] Because the current law concerning whether a party infringes another's patent tends to be indeterminate in many instances—even when all of the relevant facts are known precisely—whether *B*'s actions are "privileged" (i.e., *B* does not infringe) or instead violate a "duty" (i.e., *B* does infringe) can typically be predicted only in a probabilistic sense before final judgment.[243]

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[236] *See* TRAVIS NORSEN, FOUNDATIONS OF QUANTUM MECHANICS: AN EXPLORATION OF THE PHYSICAL MEANING OF QUANTUM THEORY (2017) (describing a variety of proposed explanations of the quantum measurement problem).

[237] *See id.* at 59–69 (concisely describing the measurement problem).

[238] *See generally* RESTATEMENT (SECOND) OF JUDGMENTS (AM. L. INST. 1982) (summarizing the law relating to the legal system's production of valid, binding, and final judgments).

[239] *See generally* RESTATEMENT (FIRST) OF JUDGMENTS § 69 cmt. a (AM. L. INST. 1942) ("If a judgment rendered by a court of first instance is reversed by the appellate court and a final judgment is entered by the appellate court or by the court of first instance in pursuance of the mandate of the appellate court, this latter judgment is conclusive between the parties.").

[240] Note that the probabilistic nature of legal relations and the resulting binary judgment that selects either a Hohfeldian right or duty is not essential. Scholars have posited the possibility of probabilistic judgments, in which the usual remedy is only partially enforced. Ian Ayres & Paul Klemperer, *Limiting Patentees' Market Power Without Reducing Innovation Incentives: The Perverse Benefits of Uncertainty and Non-Injunctive Remedies*, 97 MICH. L. REV. 985, 1029–31 (1999) (proposing a regime in which successful patentholders receive only a percentage of the total damages otherwise awarded). For simplicity—and to align the discussion with that of measurements in quantum mechanics—I assume that judgments can only be binary. However, like the probabilistic model of legal entitlements, Hohfeld's approach could be extended to probabilistic judgments.

[241] *See, e.g.*, Schering Corp. v. Mylan Pharms., Inc., No. 09-6383, 2011 WL 2446563, at *1 (D.N.J. June 15, 2011) (describing patents-in-suit as covering pharmaceutical drugs that lower cholesterol).

[242] *See* John M. Golden, *"Patent Trolls" and Patent Remedies*, 85 TEX. L. REV. 2111, 2130–31 (2007) (examining patent "design-arounds").

[243] *See* Mark A. Lemley & Carl Shapiro, *Probabilistic Patents*, 19 J. ECON. PERSPS. 75 (2005); Indeed, it appears patent rights have been uncertain for many years. *See, e.g.*, *E. Bement & Sons*

Thus, before a final judgment, whether Company *A* has a *strict-right* to prevent Company *B* from undertaking its "design around" to the patent can be modeled as a superposition of probability states analogous to a superposition of quantum spin states.[244] If we let "$|j\rangle$" represent the quantum state of the specific legal relation at issue, then:

$$|j\rangle = a\ |j_r\rangle + b\ |j_{\neg r}\rangle \tag{29}$$

In this formula, $|j_r\rangle$ is a *strict-right* and $|j_{\neg r}\rangle$ is a *no-right*.[245] In terms of a legal proposition, the quantum legal relation $|j\rangle$—here $|j\rangle_1$, to represent the first-order nature of the legal relation—would substitute for the classical first-order relation, j_1 (or $j_{\neg 1}$) that appear in (14).[246] In other words, a first-order "quantum" legal proposition is as follows:

$$|J_1\rangle = X_{\ |j\rangle_1}\ Y(S_1) \tag{30}$$

Prior to any final judgment, Company *A*'s relevant legal relation exists in a probabilistic superposition of a *strict-right* and a *no-right*. The probabilities if Company *A* goes to court to request a judgment regarding the applicable relation, i.e., whether it will be found to have a *strict-right* or *no-right*, respectively are:[247]

$$P(j_r) = |a|^2 \text{ and } P(j_{\neg r}) = |b|^2, \text{ where } |a|^2 + |b|^2 = 1 \tag{31}$$

Because $|j_r\rangle$ and $|j_{\neg r}\rangle$ represent vectors, it is possible to display a first-order legal relation as a vector on a graph.[248] If we assume, for simplicity,

v. La Dow, 66 F. 185, 190 (C.C.N.D.N.Y. 1895) ("[N]o property is so uncertain as 'patent rights'; no property more speculative in character or held by a more precarious tenure. An applicant who goes into the patent office with claims expanded to correspond with his unbounded faith in the invention, may emerge therefrom with a shriveled parchment which protects only that which any ingenious infringer can evade. Even this may be taken from him by the courts. Indeed, it is only after a patentee has passed successfully the ordeal of judicial interpretation that he can speak with any real certainty as to the scope and character of his invention.").

[244] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 7–8 (describing legal relations that exist in a probabilistic superposition of states prior to judgment). *See generally* Dan L. Burk & Mark A. Lemley, *Quantum Patent Mechanics*, 9 LEWIS & CLARK L. REV. 29, 31 (2005) ("The problem may be worse than a simple failure to acknowledge subconscious decisions that affect the scope of a patent, however. This indeterminacy may well be inherent in the process of mapping words to things, as modern literary theorists suggest.").

[245] *See supra* notes 121–122 (noting that *r* denotes a strict-right and ~r denotes a no-right).

[246] *See supra* note 164 (setting forth a formal notation for a classical first-order legal proposition).

[247] *See* GREINER, *supra* note 210, at 333–36.

[248] B. HAGUE, AN INTRODUCTION TO VECTOR ANALYSIS FOR PHYSICISTS AND ENGINEERS 4 (D. Martin ed., Springer 2012) (1939) (discussing the graphical representation of vectors).

that *a* and *b* are real numbers,[249] then $|j\rangle$ will be the vector (*a*, *b*) subject to the condition in equation (31).[250] (See Fig. 10.) Thus, unlike the classical first-order relations, which were either (1, 0) (for r_1) or (0, 1) (for $\sim r_1$), the post-classical relations can take on any value in two-dimensional abstract space.[251]

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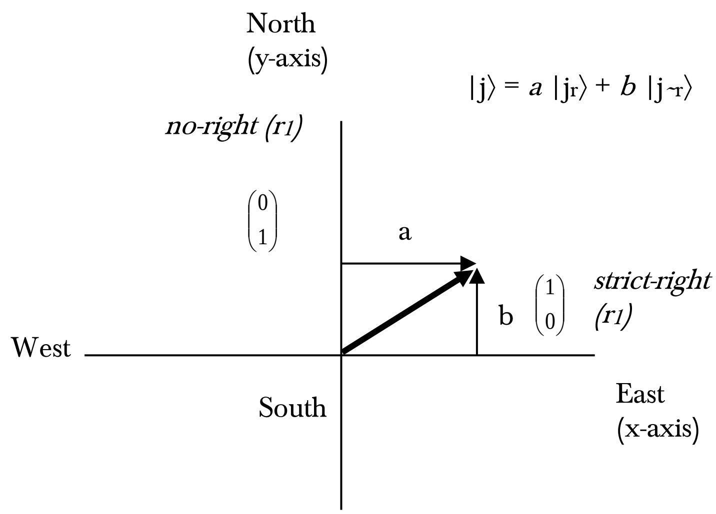
[249] Generally, *a* and *b* will be real numbers when there are no "interference" effects of the legal relation upon itself or with other legal relations. *See generally* Mohammed Sanduk, *Is There a Physical Reason Beyond the Imaginary* i *in the Quantum Mechanics Formulation?*, 5 INT. J. QUANTUM FOUNDS. 69 (2019) (examining the possible reasons for the use of imaginary numbers in quantum theory). As alluded to earlier, interference in quantum mechanics occurs when the evolution of a quantum state vector alters *a* and *b* in non-classical ways that involve the interaction between the "eigenstates" associated with *a* and *b*. V. MURUGAN, QUANTUM MECHANICS 22 (2014) (describing the non-classical nature of quantum interference).

[250] Note, however, that while precise values for *a* and *b* may exist theoretically, in practice, it will often be very difficult, if not impossible, to predict these values with precision. A more practical formalism might view *a* and *b* as expectation values from a given probability distribution. However, such an extension would not materially change the underlying formalism presented here. *See, e.g.*, GÉZA SCHAY, INTRODUCTION TO PROBABILITY WITH STATISTICAL APPLICATIONS 127–28 (2007) (explaining how to derive expectation values from probability distributions).

[251] In the event one sought to model quantum-like interference effects among Hohfeldian relations, which would entail *a* and *b* taking on complex values, *see supra* note 249, a three-dimensional Bloch sphere would be more appropriate than the two-dimensional presentation offered here. However, given that Figure 10 abstracts away from such effects, leaving *a* and *b* as real numbers, a two-dimensional approach is suitable. *See* RAY LAPIERRE, INTRODUCTION TO QUANTUM COMPUTING 54–55 (2021) (describing the Bloch sphere and implicitly showing that when there is no interference due to the lack of a phase difference in a qubit (ϕ=0), a two-dimensional representation is sufficient).



Unlike the classical first-order right relations, which are merely one of two vectors, a post-classical relation, $|j\rangle$, is the superposition of two classical relations, $|j_r\rangle$ (*strict-right*) and $|j_{\neg r}\rangle$ (*no-right*). When a and b are real numbers, the vector $|j\rangle$ is simply (a, b). The magnitude of $|j\rangle$, $|a|^2 + |b|^2$, is always 1, because the probability of measuring $|j\rangle$ in one of the two classical states must add up to 100%. When $|j\rangle$ is actually measured upon a final judgment, it instantaneously collapses into one of the two vectors representing the classical states, i.e., either (1, 0) or (0, 1), which represent r_1 (*strict-right*) and $\sim r_1$ (*no-right*), respectively.

Fig. 10. A Two-Dimensional Post-Classical Representation of a Superposition of Strict- and No-Right Vectors

If Company *A* goes to court to adjudicate whether Company *B*'s actions infringe its patents—assuming the parties do not settle the case, appeals are exhausted, and procedural formalities are met—then a final judgment will issue from the court.[252] This judgment will be that either Company *B*'s actions infringe the patent or they do not, but nothing in

[252] *See supra* note 239 and accompanying text (describing the result of a final judgment).

between.[253] Thus, the result of a judgment—like a measurement in quantum mechanics—is to "collapse" the probabilistic superposition of legal relations into a single classical state,[254] here *either* a *strict-right or* a *no-right*.[255]

ii. A Probabilistic Superposition of Nth-Order Hohfeldian Relations

Whether a legal actor holds a *power* can also be indeterminate prior to judgment.[256] For example, Section 5 of the Fourteenth Amendment of the United States Constitution provides Congress with the "power to enforce, by appropriate legislation, the provisions of" of that amendment.[257] A number of cases in the U.S. Supreme Court have addressed the issue of whether Congress exceeded the scope of this power by purporting to pass legislation pursuant to Section 5, including legislation to protect voting rights and religious freedom, and the outcomes in these cases have been far from clear before the fact.[258]

The same equations for the first-order relations apply to powers other than for the fact that the superposition is one of two second-order relations (a *power* and a *disability*):[259]

$$|j\rangle_2 = a_2\,|j_r\rangle_2 + b_2\,|j_{\neg r}\rangle_2 \tag{32}$$

[253] *See* Andrew D. Selbst, *An Institutional View of Algorithmic Impact Assessments*, 35 HARV. J.L. & TECH. 117, 131 (2021) ("A judgment of liability is a binary question. A person is either at fault or not; a plaintiff wins or a defendant does.").

[254] *See* Kenney Hegland, *Indeterminacy: I Hardly Knew Thee*, 33 ARIZ. L. REV. 509, 520 (1991) (analogizing judgment to quantum measurement to explain how "theoretically hard cases becom[e] easy at the point of decision").

[255] Alternatively, if one desires to retain the notion that all legal relations are knowable in principle, a judgment can still be viewed in this model as simply "unveiling" the judge's (or jury's) view of the "proper" result in any given case. The mathematical formalism presented here remains nearly the same, with the probabilities representing the "best knowable guess" as to judgment outside the judge's chambers (or jury's box). *See supra* notes 198–207 and accompanying text.

[256] *See* MICHAEL J. PERRY, THE CONSTITUTION, THE COURTS, AND HUMAN RIGHTS 41 (1982) (stating that the powers delegated to regulatory agencies are "quite indeterminate in scope").

[257] U.S. CONST. amend. XIV, § 5.

[258] *See, e.g.*, Richard H. Fallon, Jr., *The "Conservative" Paths of the Rehnquist Court's Federalism Decisions*, 69 U. CHI. L. REV. 429, 456 n.167 (2002) (contending that "[t]he uncertainty arose partly because the [Supreme] Court's opinion construing Congress's power under Section 5 of the Fourteenth Amendment in *Katzenbach v Morgan* . . . blended a puzzling mix of rationales").

[259] *See supra* notes 142–148 and accompanying text (formalizing second-order legal propositions).

Here, $|j\rangle_2$ is the state of the power relation, where the subscript "2" indicates that it is a second-order relation.[260] As such, $|j_r\rangle_2$ is a *power* and $|j_{\neg r}\rangle_2$ is a *disability*.[261] Again, the probability that a court will find that a power exists is $|a_2|^2$ and that no power exists, $|b_2|^2$, where $|a_2|^2 + |b_2|^2 = 1$.[262] (The subscripts attached to *a* and *b* indicate the order of the relation they multiply, here "2" for second-order.)

The "ket" notation here, namely the $|\rangle$ symbol around the legal relation indicates that this is the quantum, rather than the classical, version of the Hohfeldian relations.[263] Notably, unlike in quantum mechanics, where the state of the system is generally a vector in Hilbert space, as discussed further below, the most general state of a legal relation is a tensor in Hohfeldian space, which is only a vector for first-order relations.[264]

In general, any *n*th order legal relation as it exists before judgment can be expressed as:[265]

$$|j\rangle_n = a_n|j_r\rangle_n + b_n\,|j_{\neg r}\rangle_n \text{ where } P(j_r)_n = |a_n|^2 \text{ and } P(j_{\neg r})_n = |b_n|^2, \text{ and } |a_n|^2 + |b_n|^2 = 1 \qquad (33)$$

Once a court makes a judgment on whether an actor possesses a power or not, the quantum state expressed by (33) collapses to the classical state expressed above by (13).[266]

An *n*th-order "quantum" legal proposition—that is, a statement involving a legal relation between two legal actors with respect to some

[260] *See id.*

[261] *See id.* (noting that $\sim p$, which is equivalent to $\sim r_2$, is the absence of legal power).

[262] *See supra* note 247 and accompanying text (setting forth probability rule).

[263] *See supra* notes 244–251 and accompanying text (describing the first-order quantum Hohfeldian relations).

[264] *See infra* notes 265–277 and accompanying text (describing the nth-order quantum Hohfeldian relations).

[265] *See supra* notes 152–158 (formalizing nth order legal propositions). Again, the coefficients a_n and b_n may in general be complex numbers, but legal relations could more simply be modeled by real coefficients. As noted earlier, complex coefficients are used to reflect the full power of the formalism, particularly its application back to model quantum mechanics and the measurement problem. *See infra* Part VI.B.

[266] Such a collapse is effectuated by the court exercising a second-order power to instantaneously "rotate" the legal state vector to one of the two Hohfeldian outcome positions, that is either power (right) or disability (no-right) but not a superposition of both states. This second-order judicial power—which applies even when statutes, rules, or doctrines otherwise "constrain" judges so as not to be a form of common-law adjudication—is a limited form of the power exercised by legislatures in enacting law. *Cf.* Frederick Schauer, *Do Cases Make Bad Law?*, 73 U. CHI. L. REV. 883, 887–88 (2006) (espousing that it is "far too late in the day to deny that judges are often (some would say 'always') engaged in the process of making law").

lower-order legal relation or state of affairs[267]—would be expressed as follows:

$$|J_n\rangle = (X_{|j\rangle_n} Y(S_n))(\, |J_{n-1}\rangle) \text{ for } n > 0 \quad (34)$$[268]

The general quantum legal proposition in (34) is very similar to the general classical legal proposition presented earlier (13), with the fundamental difference that (34) is a probabilistic—rather than certain—proposition.[269] Moreover, the *power* legal relations $|j_r\rangle_n$ ($n > 1$) in the quantum legal proposition are unlike their classical counterparts, which can only change a *strict-right* to a *no-right*, or vice-versa.[270] Rather, a post-classical nth-order power may, in the most general case, be represented by a "unitary" tensor operator of rank 2^{n-1}, which can "rotate" the 2^{n-1} components of an $(n-1)$-th legal relation state *arbitrarily* within an $(n-1)$-th dimensional complex "Hohfeldian" vector space.[271] For simplicity, one can generally split any higher-order power tensor apart into multiple second-rank tensors, which multiply each other to transform the lower-order legal relation.[272] Moreover, given the specific nature of the Hohfeldian legal relations, as with the classical powers and disabilities, one can represent any third-order and higher power as a second-rank tensor

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[267] *See supra* Part II.A (explaining the difference between legal relations and legal propositions).

[268] Here, the classical existence bit becomes an existence "qubit," $|J_0$, such that the existence (or not) of a legal proposition may be in a probabilistic superposition prior to a final judgment by a Court. For instance, whether a foreign national on foreign soil is subject to a given body of U.S. law in general—as opposed to some specific first-order obligation—could be viewed as probabilistic "existence" question.

[269] *See supra* note 158 and accompanying text (setting forth an n-th order legal proposition).

[270] *See supra* note 176 and accompanying text (describing a classical mathematical formalism for a Hohfeldian second-order power).

[271] Such an operator will only effectuate *changes* in lower-order legal relations. As noted above, a complete description requires "creation" and "termination" operators that create and destroy legal relations via the existence bit. *See supra* note 268 and accompanying text (describing the quantum existence bit).

[272] *See* BORISENKO & TARASOV, *supra* note 169, at 76–77 (describing higher-order tensors). Specifically, if we let U be a power operator of order n and rank 2^{n-1}, and R be legal relation of order n-1 and rank 2^{n-2}, then R can be transformed into a new relation R′ as follows:

$$R'_{ij\,k\ldots l} = U_{ip}U_{jq}U_{kr} \ldots U_{ls}R_{pqr\ldots s}$$

where U = $U_{ip}U_{jq}U_{kr} \ldots U_{ls}$ and runs over 2^{n-1} indices and where R′ and R run over 2^{n-2} indices. Each second-rank operator U_{ip} describes the transformation of R in one dimension of the applicable Hohfeldian space. *See id.* Note that this operation is an active transformation of the lower-rank tensor representing the lower-order tensor, rather than a coordinate transformation, which is similar in form. *See generally* BRENT ADAMS, SURYA KALIDINDI & DAVID FULLWOOD, MICROSTRUCTURE SENSITIVE DESIGN FOR PERFORMANCE OPTIMIZATION 35–38 (2012) (explaining the difference between rotation tensors and coordinate transformations).

that transforms the elements of any lower-order power, also second-rank tensors, in the desired fashion.[273]

In this regard, a second-order power, $|j_r\rangle_2$, can in principle rotate a first-order legal relation anywhere within a two-dimensional Hohfeldian space.[274] For example, in the patent hypothetical above, a new law passed by Congress may instantaneously update the probability that Company *B*'s actions would be found to escape liability.[275] In this event, Congress's exercise of power changes the *a*'s and *b*'s of the relevant first-order legal relation, thereby rotating the relation's associated state vector in Hohfeldian space.[276] However, as noted earlier, the judgment power of a court is a limited second-order (or, less frequently, higher-order) power, because it can only instantaneously "rotate" the vector (or lower-order power) to one of two positions—*right* or *no-right* for first-order relations (or liability or immunity for higher-order relations).[277] In contrast to powers, the *disability* operator, $|j_{\neg r}\rangle_n$—since it has no effect on the underlying legal relation states on which it operates—is essentially the same as its classical counterpart and is represented by a second-rank identity tensor operator.[278]

IV. "PHYSICAL" PROPERTIES OF LEGAL PROPOSITIONS AND SYSTEMS

One of the significant benefits of the mathematical formalism presented above is that it naturally lends itself to a variety of quantitative measures used in physics to describe the properties of systems. As an

[273] *See generally* JACK B. KUIPERS, QUATERNIONS AND ROTATION SEQUENCES: A PRIMER WITH APPLICATIONS TO ORBITS, AEROSPACE, AND VIRTUAL REALITY (2020) (describing a variety of mathematical approaches to rotations in two and three dimensions).

[274] If we represent the state of a first-order system as a qubit on a Bloch sphere, a second-order power will be represented by a universal rotation operator that can rotate the qubit vector anywhere on the Bloch sphere. *See* NIELSEN & CHUANG, *supra* note 174, at 175–76 (describing a universal rotation operator).

[275] *See, e.g.*, L.M. BROWNLEE, INTELLECTUAL PROPERTY DUE DILIGENCE IN CORPORATE TRANSACTIONS § 5:102 (2022) ("Effective September 11, 2011, the AIA added 35 U.S.C.A. § 298, which makes it more difficult to prove willful infringement . . . .").

[276] *See* STORRS MCCALL, THE CONSISTENCY OF ARITHMETIC AND OTHER ESSAYS 190 (2014) (noting that as the state vector rotates in Hilbert space, "up/down probabilities in each case will be altered").

[277] *See supra* notes 255, 266 and accompanying text (explaining that a judgment power "collapses" the state vector to one of two positions—right or no-right for first-order relations or power or disability for higher-order relations). Note that this rotation does not move the vector through all of the intermediate positions from initial to final state, but instead instantaneously transports from the vector from the original to final state.

[278] *See supra* note 185 and accompanying text (describing an *n*th-rank identity tensor).

example of these measures, in this Part I provide a more conceptual description of a quantitative model of legal entropy and temperature that recently appeared in a scientific journal.[279] Although scholars have used these and other scientific and mathematical terms to describe the properties and evolution of legal systems in a metaphorical sense,[280] the approach described here allows for a precise quantitative measure of these legal properties.[281] Using the proposed measures, I briefly discuss how these measures can suitably be applied to describe the notion of "modularity" in the law.[282]

A. Legal Entropy

A legal relation may be perfectly known (e.g., a 100% chance of *strict-right* (liability) and 0% chance of *no-right* (no liability) upon judgment) or maximally indeterminate (e.g., a 50/50 chance of the same).[283] Any indeterminacy will contribute to the system's overall information entropy, a measure of legal indeterminacy of the underlying relations in the system.[284]

Specifically, the legal information entropy for a single nth-order relation can be expressed by the formula for Shannon information entropy:[285]

$$\mathrm{H}(|\mathrm{J}\rangle) = -\Sigma\, p_i \log_2 p_i$$

$$= -\mathrm{P}(\mathrm{j_r})_\mathrm{n} \log_2 \mathrm{P}(\mathrm{j_r})_\mathrm{n} - \mathrm{P}(\mathrm{j_{\neg r}})_\mathrm{n} \log_2 \mathrm{P}(\mathrm{j_{\neg r}})_\mathrm{n}$$

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[279] *See infr*a Part IV.A–B (describing legal entropy and temperature).

[280] *See supra* note 40.

[281] *See infra* Part IV.A–B; *see also* Sichelman, *Legal Entropy*, *supra* note 21, at 5–12 (proposing a mathematical model of legal entropy). As I note in *Legal Entropy*, several scholars have used the term "legal entropy" to refer to a measure of textual ambiguity or similar properties in legal texts that are quite different from the broad notion of quantitative legal entropy offered in that article and described here. *See id.* at 3–4.

[282] *See infra* Part IV.C (describing legal entropy and temperature in the context of legal modularity). *See generally* Henry Smith, *Property as the Law of Things*, 125 HARV. L. REV. 1691 (2012) (explaining the role of modularity in the context of property law).

[283] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 8–9.

[284] *See id.* at 4–7 (describing the various types of entropy present in legal systems).

[285] *See id.* at 5–6 (explaining how the Shannon entropy formula measures legal entropy). *See generally* MARINESCU & MARINESCU, *supra* note 165, at 232–41 (discussing Shannon entropy from a general information theory perspective). Note that "while the Von Neumann entropy—which in effect measures the indeterminacy of a mixed quantum state with respect to its entangled substates—is zero for a pure quantum state, there is nonetheless Shannon information entropy for a pure state with respect to the indeterminacy of its potential measurement outcomes prior to a measurement." Sichelman, *Legal Entropy*, *supra* note 21, at 8 n.20.

$$= -P(j_r)_n \, log_2 \, P(j_r)_n - (1-P(j_r))_n \, log_2 \, (1-P(j_r))_n \quad (35)$$ [286]

If one plots this graph for values of $P(j_r)$, the result is an inverted parabola (see Fig. 11).[287]

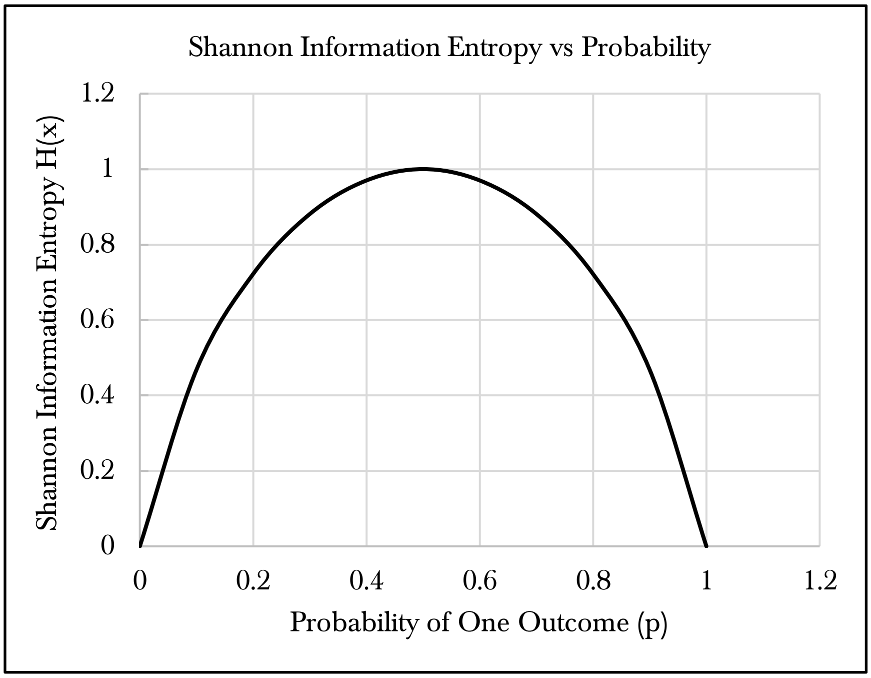

Fig. 11. Binary Shannon Entropy vs. Probability of an Event Occurring[288]

For instance, if there is a 50/50 chance of a defendant being found guilty, then the informational entropy is $-2(0.5) \, log_2 \, (0.5) = 1$.[289] In Shannon terms, this means that there is one bit of "informational indeterminacy" regarding the outcome of a judgment.[290] When the chances are skewed (e.g., 70/30), there is less than one bit of indeterminacy.[291] When there is 100% chance of a given judgment, then there is no information entropy.[292] The total informational entropy of a legal system comprising independent

[286] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 8–9.
[287] *Id.* at 9.
[288] *See* ChatGPT, Plot of Shannon Information Entropy vs. Probability, OPENAI (last visited Nov.. 26, 2024).
[289] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 5–9.
[290] *See id.*
[291] *See id.*
[292] *See id.*

legal relations can be characterized simply by adding up the entropy of each relation, though depending upon the application, a more useful measure might be the average information entropy per relation.[293]

B. Legal Temperature

Physical temperature is directly related to the average kinetic energy of a group of particles, such as the average kinetic energy of a system of particles bouncing off the walls of a three-dimensional box.[294] Temperature measures a property of a particular physical system.[295] Thus, to construct legal temperature, it is necessary to define an appropriate "legal system."[296] In general, a particular legal system will be a specific set of legal propositions concerning specific legal actors and specific states of affairs of interest.[297] For instance, a legal system might include all the legal propositions concerning a particular corporation, including those concerning its directors and officers, employees, suppliers, purchasers, federal and state governments exercising authority over the corporation, and so forth.[298] These legal propositions may be roughly divided into subsystems, such as between those propositions that affect legal actors "inside" the corporation and "outside" the corporation.[299] In this manner, one can compare legal "temperatures" from one subsystem to the next.[300]

Unlike Shannon entropy, there is no well-accepted measure of information "temperature" or information "energy."[301] Nonetheless, one

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[293] *See id.* at 9–10 (discussing systemwide legal entropy).

[294] *See* PETER EASTMAN, INTRODUCTION TO STATISTICAL MECHANICS § 4.1 (2015), https://web.stanford.edu/~peastman/statmech/ ("Temperature is a measure of average kinetic energy.").

[295] *See id.* (discussing systems composed of microscopic particles).

[296] *See id.* (examining a "system of interest" in contrast with a general, external "heat bath").

[297] *See supra* Part II (describing the Hohfeldian formalism of specific legal relations inhering between specific legal actors).

[298] *Cf.* Henry N. Butler & Larry E. Ribstein, *Opting Out of Fiduciary Duties: A Response to the Anti-Contractarians*, 65 WASH. L. REV. 1, 7 (1990) ("The contractual theory of the corporation states that the corporation is a set of contracts among the participants in the business, including shareholders, managers, creditors, employees and others.").

[299] *See, e.g.*, Edward B. Rock & Michael L. Wachter, *Islands of Conscious Power: Law, Norms, and the Self-Governing Corporation*, 149 U. PA. L. REV. 1619, 1697–98 (2001) ("The boundaries of the firm are thus a demarcation or jurisdictional line distinguishing relationships in which disputes are resolved by third-party enforcement from relationships that are intended to be self-enforced within the firm.").

[300] *See* WASSIM M. HADDAD, A DYNAMICAL SYSTEMS THEORY OF THERMODYNAMICS 210 (2019) (discussing subsystem temperatures).

[301] Octav Onicescu has proposed a measure of "information energy," which is useful in many contexts, but it is not exactly analogous to physical energy and has not gained wide acceptance.

can construct a rough conceptual approach that provides some insight into how these concepts might operate in legal systems. In this regard, it is useful to return to the original concept of entropy in thermodynamics.[302] Specifically, Clausius defined the increase (or decrease) of physical entropy, ΔS, in a reversible process involving an ideal engine as the ratio of the transfer of heat, ΔQ, from a heat source at a certain temperature, T, to the ideal engine.[303] Thus, if a system is relatively cold and a large amount of heat is added to it, then S increases substantially.[304] On the other hand, if a system is relatively hot, and the same amount of heat is added to it as with the cold system, the entropy of the hot system will increase much less than the cold system.[305]

As legal "heat" enters a legal system, the legal entropy of the system increases by increasing the underlying indeterminacy in outcome.[306] For instance, increasing uncertainty in the underlying facts of a given case constitutes legal "heat" that generally increases the legal entropy of the system.[307] In mathematical terms, legal heat – and, in turn, increasing legal temperature – shifts the Hohfeldian state vector away from relatively certain to relatively random states.[308] Increasing heat will maximize entropy when the expected positions of the state vectors of a legal subsystem are consistent with purely random outcomes, such that judgment of the underlying legal states is a coin flip (50/50).[309]

In line with Clausius's definition of entropy, introducing legal heat into a system will have less effect the higher the legal temperature.[310] For instance, suppose a legal subsystem's Hohfeldian state vectors are in very "rapid motion," changing outcome probabilities randomly every minute.[311] For disputes within the domain of this legal subsystem, a legal

See Mojtaba Alipour & Afshan Mohajeri, *Onicescu Information Energy in Terms of Shannon Entropy and Fisher Information Densities*, 110 MOLECULAR PHYSICS 403 (2012) (relating Onicescu information energy to Shannon information entropy).

302 *See* Sichelman, *Legal Entropy*, *supra* note 21, at 2–3.

303 *See id.*

304 *See id.*

305 *See id.*

306 *See id.* at 11.

307 *See id.*

308 *See* HUGH D. YOUNG ET AL., UNIVERSITY PHYSICS 671 (Australian ed. 2010) ("Adding heat to a body increases its disorder because it increases average molecular speeds and therefore the randomness of molecular motion.").

309 *See* Sichelman, *Legal Entropy*, *supra* note 21, at 11.

310 *See id.* at 2–3, 11.

311 *See id.* Here, the "rapid" legal "motion" of the state vectors is analogous to the rapid motion of particles in the sense that the probabilities within some legal subsystem are sufficiently fluctuating across the entire probability space—because of changing law, difficult- or impossible-to-determine relevant facts, massive variation in outcome among different decisionmakers, and

decisionmaker in effect must randomly choose the outcome, and the introduction of more heat cannot increase the entropy of the system.[312] In essence, it is as if the adjudicator—the judge or jury—must flip a coin to determine the outcome of a dispute, because the applicable doctrine or facts are so uncertain it is of no merit examining them.[313] Thus, increasing uncertainty in the underlying law or facts will have no effect on the ultimate outcome and cannot increase the subsystem's legal entropy.[314]

C. *The Relation of Temperature and Entropy to Legal "Modularity"*

Information costs "include the costs of generating information about rights in the process of delineating and publicizing them, as well as the costs incurred by third parties in processing information about the scope, nature, and validity of those rights."[315] The temperature and information entropy of a legal system are often a direct indicator of information costs. Specifically, as the relative randomness (temperature) of legal relations within a system increases, it will become more costly for actors subject to those relations to keep track of them—particularly, in determining how those relations bear upon the legality (or not) of their primary behavior.[316] Additionally, as the indeterminacy (entropy) in those relations increases, determining the scope of those relations will become more costly.[317] Thus, legal temperature and entropy can help provide a quantitative measure for the information costs—which may not merely be economic—borne by legal actors in a given legal system.[318]

Henry Smith has posited that information costs play an integral role in the "modularity" of legal systems—namely, the use of "boundaries" in the law (be they spatial or intangible) to "economize on information costs"

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other factors—that the subsystem is generally unstable, and thus highly uncertain, from an adjudicatory perspective. *See, e.g.*, Donald I. Baker & William Blumenthal, *Ideological Cycles and Unstable Antitrust Rules*, 31 ANTITRUST BULL. 323, 323–25 (1986) (noting that "[b]usinessmen cannot effectively structure their affairs and plan for the future if antitrust policy repeatedly caroms off ideological poles").

[312] See Sichelman, *Legal Entropy*, *supra* note 21, at 11.

[313] *See id.*

[314] *See id.*

[315] Henry E. Smith, *Exclusion and Property Rules in the Law of Nuisance*, 90 VA. L. REV. 965, 970–71 (2004) [hereinafter Smith, *Exclusion and Property Rules*].

[316] *See* Baker & Blumenthal, *supra* note 311, at 334 (describing a variety of costs associated with unstable legal rules).

[317] *See id.*

[318] *See* Sichelman, *Legal Entropy*, *supra* note 21, at 3–11 (offering a variety of quantitative measures for legal entropy and the related uncertainty present in legal systems).

by "hiding" classes of information "behind" the boundaries.[319] For instance, in real property, the boundary of a piece of land effectively hides the owner's (unspecified) interests in using the land from legal consideration in the investigation of actions by a third party.[320] In other words, to determine if a third party unreasonably interfered with the owner's interests, instead of examining whether a particular action on the part of the third party interfered with particular uses of the owner, we assume that when a third party unjustifiably crosses the boundary, an interference occurs.[321] This assumption economizes on information costs by using the boundary as a reliable proxy for actual interference with the owner's specific interests.[322]

Indeed, in separate work, Smith and I quantify the amount of modularity in a given legal system by constructing a Hohfeldian network of legal relations among legal actors.[323] For instance, in Figure 12 below, there are two plots of land separated by a commons.[324] One plot is owned by Y and another owned by C. By depicting the Hohfeldian legal relations, rights, privileges, and the like, as edges (vectors) between the two owners as well as between the owners and third-parties, we construct a complex Hohfeldian graph of nodes (legal actors) and directed edges (legal relations).[325] The vectors connecting the nodes of the network adhere to the mathematical formalism described earlier.[326] Specifically, classical vectors operate as definite rights (or no-rights) and probabilistic vectors may be weighted accordingly to take into the indeterminate nature of the underlying legal relation.[327]

By adapting the rich mathematics of network theory from the social sciences, we show that one can calculate a numerical measure of modularity of legal systems and subsystems.[328] A fully modular system is completely decomposable into separate units with bright-line legal

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[319] *See* Henry E. Smith, *On the Economy of Concepts in Property*, 160 U. PA. L. REV. 2097, 2115–16 (2012) [hereinafter Smith, *Property*].

[320] *See id.* ("By setting up cheap and rough proxies like boundary crossings, property law can indirectly protect a wide range of largely unspecified interests in use, the details of which are of no particular relevance to those under a duty to respect the right . . . .").

[321] *See id.*

[322] *See id.*

[323] *See* Ted M. Sichelman & Henry E. Smith, *A Network Model of Legal Relations*, 382 PHIL. TRANSACTIONS ROYAL SOC'Y A 20230153 (2024).

[324] *See id.* at 6–8 (setting forth an example of a Hohfeldian property network).

[325] *See id.*

[326] *See supra* Part III (proposing a mathematical formalism for classical and quantum Hohfeldian relations).

[327] *See* Sichelman & Smith, *supra* note 323323, at 20–21.

[328] *See id.* at 11–15.

boundaries around the subsystems of interest (e.g., an owned plot) while a non-modular system is do densely connected that legal boundaries become meaningless.[329]

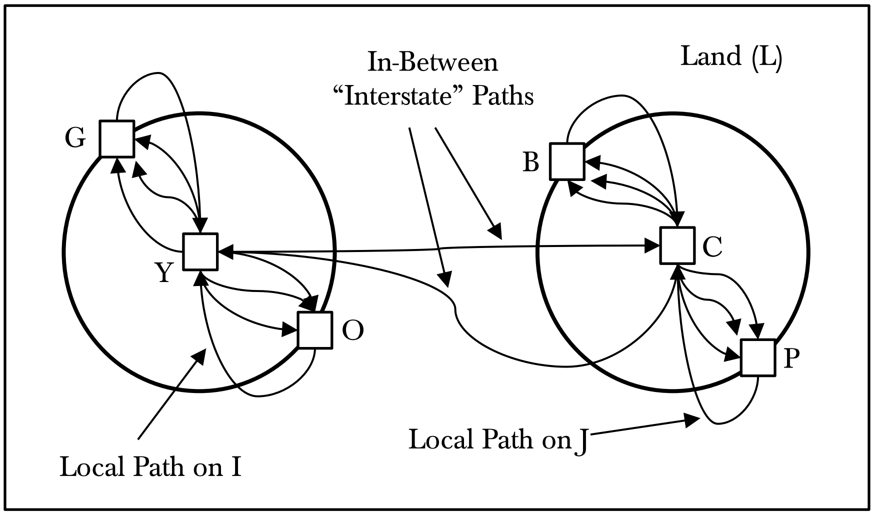

Fig. 12. Two Subplots, a Commons, and Legal Actors Connected by Partially Modular Hohfeldian Legal Relations on a Parcel of Land

Of course, erecting boundaries as proxies can introduce error costs in allocating rights and duties, so it is important to place some constraints on the modularization of law.[330] The notions of legal temperature and entropy can play an important role in imposing such constraints. Namely, it is only when the legal system inside the boundary has relatively low temperature and entropy, especially when compared to the temperature and entropy near or outside the boundary, that modularity will suitably serve its role to reduce information costs without imposing significant error

[329] *See id.*

[330] *See id.* at 19–20 ("When information costs are introduced back into the system, the reduction in information costs gained from exclusionary approaches—which modularize otherwise individuated laws—must be weighed against the increase in error costs.").

costs.[331] In the case of real property, this condition arguably holds well.[332] Although an owner may be extremely active in using the land, the legal relations governing that use will tend to be stable and certain.[333] In this instance, an owner will generally be free to use the land in a variety of manners, and any restraints on such use will tend to be few, knowable, and unchanging.[334]

Near the boundary, however, legal relations may be regularly shifting—thereby increasing legal temperature—as third parties proffer various justifications for encroaching upon the owner's land.[335] To the extent it is difficult to predict whether these justifications will pass muster in court, legal relations near the boundary exhibit a high degree of entropy.[336] In other words, for real property, the bulk of the action lies at the boundary, whereas inside (and indeed, well outside) the boundary, temperature and entropy are relatively low.[337] If, on the other hand, entropy and temperature were to rise inside the boundary—for instance, as the result of significant, ever-changing, and indeterminate governmental regulation regarding the uses that the owner could undertake—then the modularity of, in Smith's terms, an "exclusionary" approach to property becomes less attractive, instead yielding to a more particularized "governance" approach.[338]

There are similar principles at work in theory of the firm.[339] Inside the corporation's "boundary," legal temperature and entropy are very likely low due to dense and stable contracting, especially compared with legal activity outside the firm's boundary vis-à-vis third-party contracts and

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[331] To be certain, although such a condition is generally necessary to reduce error costs, it is not always sufficient to do so, because there may be certain, knowable, and frequent uses inside the boundary that potentially violate applicable obligations to the State or third parties and need to be analyzed on a case-by-case basis. *See id.* (noting that when there is no modularity present in a Hohfeldian network that the system entails a "governance" regime, which tends to analyze individual uses rather than stable boundaries to determine the violation of a legal duty).

[332] *See id.* at 11–15 (determining the quantitative modularity of a hypothetical property system).

[333] *Cf.* Smith, *Property*, *supra* note 319, at 2115 ("In effect, the exclusion strategy allows the property system to manage the complexity of resources uses through modularity, with much information hidden in property modules.").

[334] *See id.* at 2115–16.

[335] *See id.*

[336] *See generally* Sichelman, *Legal Entropy*, *supra* note 21, at 11 (examining the role of entropy and temperature relative to legal boundaries).

[337] *See id.*

[338] *See id.*; Smith, *supra* note 319, at 2116–17.

[339] *See supra* notes 298–300 (describing a temperature gradient across the corporate boundary).

state regulation.[340] Like real property, the legal system can economize on information costs by treating the congeries of contractual, agency, and other obligations among directors and officers, employees, shareholders, and others directly associated with the corporation in name as delineating a singular legal entity, "the corporation," that enjoys a measure of its own ontological significance.[341] Thus, while Hohfeld was correct to merely reduce corporate entities to a collection of legal relations among individual, natural persons,[342] from an information cost standpoint, we can economize on legal analysis by positing a corporation as an "emergent," independent entity.[343] Importantly, doing so is justified—that is, does not lead to substantial error costs—when the temperature and entropy of the legal relations relating to activity inside the firm's boundary are relatively low compared to that of relations concerning outside activity.

V. IMPLICATIONS OF A QUANTUM FORMALISM OF LEGAL RELATIONS

In addition to providing precise quantitative measures of the properties of legal systems, the formalism proposed here has ramifications for the nature of legal interpretation, legal artificial intelligence, and game theory and the law. I briefly address each area in turn.

A. Post-Classical Relations and the Nature of Legal Reasoning

Although Hohfeld is often considered a proto-legal realist, in many ways, his framework is decidedly formalist in nature.[344] Specifically, legal formalism—or at least the modern characterization of the approach—posits that given a well-defined legal rule and all of the facts needed to decide to whether a given set of legal actors complied (or not) with the legal rule,

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[340] *See* Henry E. Smith, *Intellectual Property as Property: Delineating Entitlements in Information*, 116 YALE L.J. 1742, 1821 (2007) ("[T]he boundaries of a firm render the nexus of contracts more thing-like and partake of some of the information-cost advantages of the exclusion strategy.").

[341] *See id.*

[342] *See* Wesley Newcomb Hohfeld, *Nature of Stockholders' Individual Liability for Corporation Debts*, 9 COLUM. L. REV. 285, 289 (1909) ("The only conduct of which the state can take notice by its laws must spring from natural persons—it cannot be derived from any abstraction . . . .").

[343] *See* Henry E. Smith, *Emergent Property*, *in* PHILOSOPHICAL FOUNDATIONS OF PROPERTY LAW 325–27 (James Penner & Henry E. Smith eds., 2013).

[344] *See* David Frydrych, *Hohfeld vs. The Legal Realists*, 24 LEGAL THEORY 291 (2018) (arguing that Hohfeld was not a legal realist).

the "correct" outcome of any legal problem is fully determined.[345] In other words, in this case, adjudication does not require resorting to policy-driven preferences or extra-legal concerns—legal outcomes are simply the application of the law to the facts.[346] For instance, if the legal rule is "A third-party shall not enter an owner's land without permission from the owner under any circumstances whatsoever," and *A* enters *L*, then *A* has breached the rule.[347]

The classical Hohfeldian framework provides a coherent theory of this formalist approach to adjudication.[348] Under this framework, a final judgment must result in a complete classical legal relation, J_n, where:

$$J_n = (X_{j_n} Y(S_n))(J_{n-1}) \text{ where } j_n = r_n \text{ or } {\sim}r_n,\ n > 0, \text{ and } J_0 = 1 \qquad (13)$$

I term the general form of (13) the "structure" of the legal relation.[349] To form a legal proposition, a judge—or a legal observer attempting to predict the outcome of an actual or hypothetical legal dispute—must "fill in" this structure with appropriate "content,"[350] i.e., particular legal actors, a

[345] *See* Mark R. Brown & Andrew C. Greenberg, *On Formally Undecidable Propositions of Law: Legal Indeterminacy and the Implications of Metamathematics*, 43 HASTINGS L.J. 1439, 1446 (1992) ("Formalism, remember, is the belief that a system will mechanically yield complete, consistent, and 'correct' results.").

[346] Thomas C. Grey, *Modern American Legal Thought*, 106 YALE L.J. 493, 495 (1996) (reviewing NEIL DUXBURY, PATTERNS OF AMERICAN JURISPRUDENCE (1995)) (stating that in "Langdellian legal theory . . . . law should be formal, producing outcomes by the application of rules to facts without any intervening exercise of discretion"); GRANT GILMORE, THE AGES OF AMERICAN LAW 62 (1977) (stating that formalism perceives the law as "a closed, logical system" in which "[t]he judicial function has nothing to do with the adaptation of rules of law to changing conditions; it is restricted to the discovery of what the true rules of law are and indeed always have been").

[347] Chris Guthrie et al., *Blinking on the Bench: How Judges Decide Cases*, 93 CORNELL L. REV. 1, 2 (2007) ("For the formalists, the judicial system is a 'giant syllogism machine,' and the judge acts like a 'highly skilled mechanic.'").

[348] *See supra* Part II.A (setting forth "classical" Hohfeldian legal propositions).

[349] *Cf.* Robert S. Summers, *How Law Is Formal and Why It Matters*, 82 CORNELL L. REV. 1165, 1178 (1997) ("[T]he structure of the rule [is] the way in which the parts of the rule are organized."); Hans Kelsen, *The Pure Theory of Law and Analytical Jurisprudence*, 55 HARV. L. REV. 44, 44 (1941) (noting that his "pure" theory of law seeks "to discover the nature of law itself, to determine its structure and its typical forms, independent of the changing content which it exhibits at different times and among different peoples").

[350] My use of "content" here is most akin to the sense of the term "argument" in computer programming that refers to an actual parameter that is the subject of an operator. *See* DICTIONARY OF COMPUTER SCIENCE, ENGINEERING AND TECHNOLOGY 22 (Phillip A. Laplante ed., 2017) (defining "argument" in one manner as the "actual parameter to a function"). Just as operators perform some operation on a particular argument to produce a particular result (e.g., if f(x) = 2x and x = 25, then 25 is the "argument" on which the operator f(x) performs its doubling operation), the structure of a legal relation "operates" on its content to produce a specific legal relation. Importantly, my use of the terms "structure" and "content" is meant to

specific Hohfeldian relation, and some particular action or set of actions (or state of the world) that is at-issue in the case.[351] In this sense, adjudication is a "synthetic" process that merges structure and content to yield an outcome reflected in a fully formed legal proposition.[352] (See Fig. 13.)

convey different—albeit, related—senses relative to the terms "form" and "substance," particularly as those terms have been used by Duncan Kennedy. *See* Kennedy, *supra* note 123, at 1687–1701 (expounding upon the "form" and "substance" of the law). For instance, Kennedy's "form" concerns whether one adopts a rule or standard and, as such, he argues that the form of a legal rule will inevitably affect its substance. *See id.* at 1712–22. Whatever the merits of such an argument, the Hohfeldian structure here admits of both rules and standards (both of which I term "rules" in the most general sense), nor does Hohfeldian structure in any sense dictate the particular content of a specific legal proposition formed from the "synthesis" of structure and content. *See infra* notes 352–357 and accompanying text.

351 *See supra* notes 120–121 and accompanying text (describing the three components of a first-order legal relation).

352 Here "synthetic" is used with weak reference to the analytic-synthetic distinction of Kant. *See* IMMANUEL KANT, CRITIQUE OF PURE REASON 48 (Norman Kemp Smith trans., MacMillian & Co. 1929) (1781). For example, the "analytic" portions of the Hohfeldian schema—namely, its structure and associated relations—are insufficient to dictate the "synthetic" process of forming specific legal propositions from intermediate legal rules (and possibly other sources). *Cf.* Peter Westen, *Poor Wesley Hohfeld*, 55 SAN DIEGO L. REV. 449, 451 (2018) ("Hohfeld's conceptions come into play *after* jurisdictions establish normative relations and, then, only for the purpose of describing them, not for the purpose of constraining them."). Notably, most of the arguments and counterarguments among legal theorists and movements, historically and today, involve synthetic aspects of the law, rather than its underlying structure, or even the content that ultimately fills that structure. More generally, nearly all—if not all—approaches to legal theory assume the Hohfeldian structure and its range of content, or some close variant thereof.

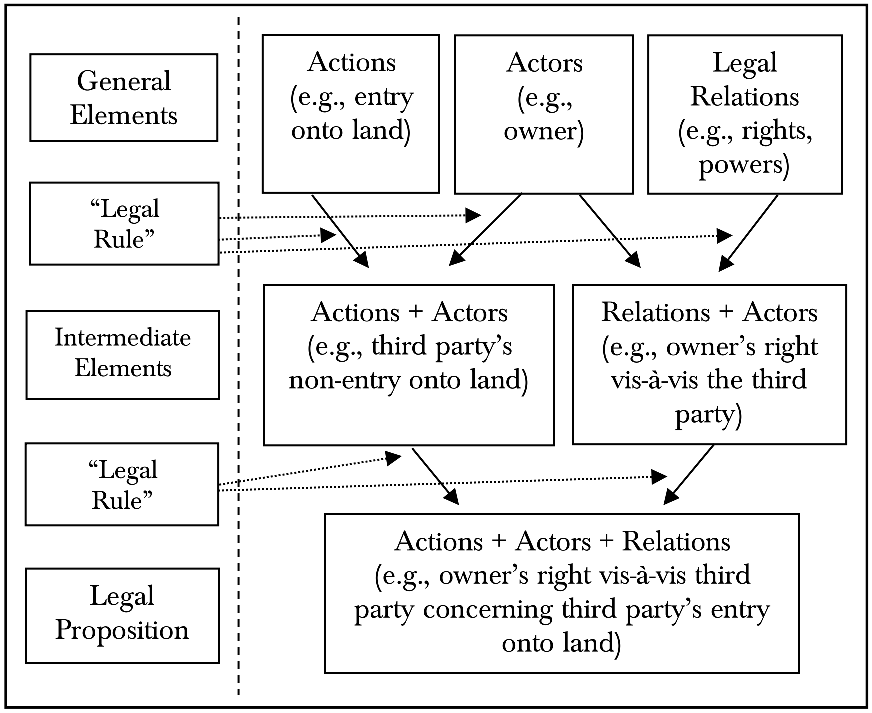

Fig. 13. The Process of Forming a Legal Proposition

Figure 13 illustrates this synthetic process. A complaint lodged by a plaintiff (e.g., an owner) typically specifies that another legal actor (e.g., a trespasser) violated a Hohfeldian duty to the plaintiff by engaging in some prohibited action (e.g., entering the owner's land without permission).[353] Thus, the complaint provides the content—the specific actions, actors, and legal relations—for the adjudication.[354] The next step is to engage in a synthetic process of merging this content with the legal structure formalized in (13).[355] The Hohfeldian structure—and the formal mathematical model associated with it—provides no indication of how a judge should engage in

[353] *See, e.g.*, Jordan v. Foust Oil Co., 447 S.E.2d 491, 498 (N.C. Ct. App. 1994) (noting that the elements of a trespass claim are "that plaintiff was in possession of the land at the time of the alleged trespass; that defendant made an unauthorized, and therefore unlawful, entry on the land; and that plaintiff was damaged by the alleged invasion of his rights of possession").

[354] Although the Supreme Court recently heightened the pleading standards for complaints, even under the previous more "liberal notice pleading allowed by the federal rules . . . the complaint [must] include the operative facts upon which a plaintiff bases his claim." Talbot v. Robert Matthews Distrib. Co., 961 F.2d 654, 660 (7th Cir. 1992).

[355] *See supra* notes 349–352 (discussing the synthetic merging of structure and content).

this process.[356] Rather, the process of finding an applicable law and interpreting it so as to yield intermediate elements and a fully formed legal proposition is "synthetic" in the very sense that it does not adhere to a structural logic.[357]

On a formalist view, however, when "well-defined" legal rules are in play, there is an independent synthetic "logic" that mandates how the application of a legal rule to a set of content yields determinate results.[358] For example, returning to the trespass example, a judge will determine that the governing legal rule is "A third-party shall not enter an owner's land without permission from the owner under any circumstances whatsoever."[359] In this event, if the owner alleges that *A* entered *L* in its complaint, a judge must necessarily form the following legal proposition:[360]

$$J_1 = O_T A(A \text{ not enter } L \text{ under any circumstances}) \qquad (39)$$

In other words, the judge must find that the owner of *L* has an absolute *strict-right* vis-à-vis *A* that *A* not enter *L*.[361] With this major premise, and the minor premise that *A* actually entered *L* without *O*'s permission, the judge must find that *A* breached his duty to *O*.[362] If the judge does not so find, the result of the adjudication is simply "incorrect" or contrary to the "rule of law."[363]

Of course, this analysis to some degree begs the question of Hohfeldian formalism by requiring that the applicable legal rule be sufficiently "well-defined."[364] Hohfeld criticized a number of ostensibly formal judicial opinions that alleged to reach their outcomes as a result of an internal legal logic, when in his view, the legal rule at issue did not fully

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[356] *See supra* note 352 and accompanying text.

[357] *See* André LeDuc, *Making the Premises About Constitutional Meaning Express: The New Originalism and Its Critics*, 31 BYU J. PUB. L. 111, 199 (2016) (remarking that "a synthetic judgment, in Kantian terms . . . cannot be reduced to a logical, syllogistic form solely on the basis of premises that state propositions expressed by the constitutional text").

[358] *See* Jack L. Goldsmith, *The New Formalism in United States Foreign Relations Law*, 70 U. COLO. L. REV. 1395, 1400 (1999) ("By 'formalistic' I simply mean that the doctrines . . . were couched in well-defined rules that did not leave much room for judicial discretion.").

[359] *See* Daniel A. Farber, *The Inevitability of Practical Reason: Statutes, Formalism, and the Rule of Law*, 45 VAND. L. REV. 533, 533–56 (1992) (assessing "plain language and related formalist methods of interpreting rules").

[360] *See supra* notes 55–74 (discussing trespass in the context of the Hohfeldian relations).

[361] *See supra* note 359 and accompanying text (describing the "formal" approach to adjudication).

[362] *See id.*

[363] *See id.*

[364] *See supra* note 358 and accompanying text.

specify the legal propositions that governed the disputes-at-hand.[365] Accordingly, Hohfeld was a proto-realist in the sense of illustrating that the law is not always the well-defined body of rules that some 19th century jurists and scholars believed it to be.[366]

But legal realism and its successor, critical legal studies, went much further in their critiques of formalist reasoning.[367] Rather, on the most radical critiques, even "well-defined" laws do not necessarily determine the outcome of legal disputes.[368] Specifically, despite the presence of complete information, such as that of our trespass hypothetical, on this view, the outcome may not be fully specified because of, for example, idiosyncratic behavior on the part of the adjudicator or the influence of extra-legal factors on the judicial process.[369] Although the formalist (or positivist) might find these factors wholly inappropriate for legal reasoning, the legal realist simply finds the law to be that which is predicted to be adjudicated.[370]

Even if one subscribes to the formalist view, from a modeling perspective, solely describing what ought to result (i.e., the "correct" outcome)—as opposed to what will result in court—has little practical value.[371] Thus, whatever one's jurisprudential take, the *post-classical* Hohfeldian structure—which takes into account unpredictability resulting from extra-legal factors (as well as the more proto-realist problems recognized by Hohfeld)—serves an important role in describing the process of adjudication.[372] In particular, instead of "law on the books" filling the gaps in the synthetic process of adjoining content to a necessary structure, "law in action" takes its place, yielding indeterminate legal relations in

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[365] *See* Hohfeld, *Fundamental Legal Conceptions*, *supra* note 14, at 35–37 (criticizing various well-known judicial opinions of the English courts).

[366] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 17 n.7 (discussing the view that Hohfeld was a proto-realist).

[367] *See, e.g.*, K. Sabeel Rahman, *Domination, Democracy, and Constitutional Political Economy in the New Gilded Age: Towards A Fourth Wave of Legal Realism?*, 94 TEX. L. REV. 1329, 1357 (2016) ("[T]he antiformalism of legal realism was more deeply developed by the critical legal studies (CLS) movement.").

[368] *See* Christopher L. Sagers, *Waiting with Brother Thomas*, 46 UCLA L. REV. 461, 462 n.5 (1998) ("According to CLS, legal rules are bad predictors of outcomes because (1) the rules, by their nature, produce unintended consequences in some cases, and (2) the rules are always in conflict with at least someone's extralegal values.").

[369] *See id.*

[370] *See* Anthony D'Amato, *The Limits of Legal Realism*, 87 YALE L.J. 468, 491 (1978) ("[F]or legal realism the 'law' is a prediction of what officials will do . . . .").

[371] *Cf.* Joseph William Singer, *Legal Realism Now*, 76 CAL. L. REV. 465, 474 (1988) ("Legal realism should be understood as the pragmatic movement in law.").

[372] *See id.* at 519 ("Their premises are more controversial than they think, and the process of generating conclusions from those premises is more indeterminate than they are willing to admit.").

many circumstances until the proverbial smash of the gavel results in a final judgment.[373]

B. From Bits to Qubits in Legal AI

Most efforts in the field of legal artificial intelligence and expert systems have been in modeling what I termed the "synthetic" process of legal reasoning.[374] Yet, the discussion provided here (and, to a large extent, by earlier deontic theorists) shows that there is a more fundamental aspect of the law—namely, its "structure"—that is amenable to modeling.[375] In the classical approach, each legal actor can be associated with a web of other actors and relevant actions of the actors (or states of the world related to the actors).[376] In this network of relationships and states of the world lie the all-important Hohfeldian relations, represented by an on/off bit (e.g., 1 for a *strict-right* and 0 for a *no-right*).[377] Law in its primary form is simply a string of 0's and 1's, defining those actions the legal actors may (and may not) perform relative to other legal actors.[378] Secondary rules specify how these primary rules come into being, change, and are terminated via Hohfeldian powers.[379]

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[373] *See generally* Joseph A. Grundfest & Peter H. Huang, *The Unexpected Value of Litigation: A Real Options Perspective*, 58 STAN. L. REV. 1267 (2006) (proposing a real options model of litigation and settlement in which uncertainty in outcome is resolved by information-revealing aspects of the litigation).

[374] *See, e.g.*, KEVIN D. ASHLEY, MODELING LEGAL ARGUMENT 223–32 (1990) (discussing a number of projects in artificial intelligence and legal reasoning); Melissa E. Love Koenig & Colleen Mandell, *A New Metaphor: How Artificial Intelligence Links Legal Reasoning and Mathematical Thinking*, 105 MARQ. L. REV. 559 (2022) (explaining how AI and logic can be used to model and inform the process of legal reasoning); Edwina L. Rissland, *Artificial Intelligence and Law: Stepping Stones to a Model of Legal Reasoning*, 99 YALE L.J. 1957, 1966 (1990) (examining the role of AI for legal reasoning and argumentation); L. Thorne McCarty, *Reflections on TAXMAN: An Experiment in Artificial Intelligence and Legal Reasoning*, 90 HARV. L. REV. 837 (1977) (same).

[375] To be certain, a subset of the AI & Law literature deals with legal "ontologies," though there has been little formal modeling of legal powers, much less an agreed-upon approach to legal relations more generally that rests upon a quantitative-mathematical model. *See supra* note 128 and accompanying text. *See generally* APPROACHES TO LEGAL ONTOLOGIES: THEORIES, DOMAINS, AND METHODS (Giovanni Sartor et al. eds., 2013) (collecting a variety of approaches to legal ontologies).

[376] *See supra* Part IV.C (describing a network model of the Hohfeldian relations).

[377] *See id.*

[378] *See id.*

[379] *See generally* HART, *supra* note 1, at 27–28 (explaining the concept of law in terms of the union of primary and secondary rules).

A properly programmed computer could store a gigantic matrix of 0's and 1's, associated actors, and associated states of the world.[380] As the state of the world changes (and as secondary rules are implemented), the matrix undergoes dynamic changes.[381] Under a post-classical approach, the bits represented 0's and 1's are replaced with "qubits" (i.e., quantum bits), which are superpositions of 0's and 1's corresponding to the superposition of legal relations existing prior to judgment.[382] Upon judgment, associated qubits would become ordinary bits, but would nearly immediately return to qubits as the "state matrix" of the legal world evolves.[383] And with qubits becoming the central unit of legal description, a quantum computer (i.e., a computer that exploits quantum mechanical phenomena) becomes the machine of choice to implement an artificially intelligent legal system.[384] By more clearly recognizing the structural inputs into the synthetic process of legal reasoning, our forays into legal artificial intelligence might progress more rapidly.[385]

C. Developing an "Endogenous" Quantum Game Theory

Game theory has become an important tool for modeling a variety of legal situations, from contracts and torts to intellectual property and public regulation.[386] Classical legal game theory implicitly assumes that the players are subject to classical legal relations—namely, that in the presence of complete information and well-defined legal rules, players either have *strict-rights* or *no-rights* with respect to the applicable set of behaviors being modeled.[387] More generally, classical game theory is limited to "classical

[380] *See, e.g.*, ASHOK N. KAMTHANE, INTRODUCTION TO DATA STRUCTURES IN C 60–62 (2002) (describing the use of matrices to store data in arrays).

[381] *See supra* Part III.A (describing changes in legal relation tensors).

[382] *See supra* Part III.B.1 (describing the use of qubits to model indeterminate legal relations).

[383] *See* R. TICCIATI, QUANTUM FIELD THEORY FOR MATHEMATICIANS 29–30 (1999) (noting how a localized quantum particle or field evolves by spreading out over time).

[384] *See generally* Mingsheng Ying, *Quantum Computation, Quantum Theory, and AI*, 174 A.I. 162 (2010) (surveying approaches to quantum artificial intelligence); Bruce M. Boghosian & Washington Taylor IV, *Simulating Quantum Mechanics on a Quantum Computer*, 120 PHYSICA D: NONLINEAR PHENOMENA 30 (1998) (describing algorithms for simulating quantum mechanical phenomena via a quantum computer).

[385] *See* Pepijn R.S. Visser & Trevor J.M. Bench-Capon, *Ontologies in the Design of Legal Knowledge Systems; Towards a Library of Legal Domain Ontologies*, *in* 99 PROC. OF JURIX (1999) ("An important reason for producing ontologies is that they form reusable building blocks for the design of (legal) knowledge systems.").

[386] *See* DOUGLAS BAIRD, ROBERT H. GERTNER & RANDAL C. PICKER, GAME THEORY AND THE LAW (1994) (applying game theory to a variety of legal fields).

[387] *See* Sichelman, *supra* note 42, at 10–15 (explaining the distinction between classical and quantum legal games).

strategy spaces," whereby a player confronted with many possible (discrete) moves at a specific decision point must choose only one of those moves when making a decision.[388] Although a player may randomize how it chooses a specific move via a "mixed strategy"—e.g., by flipping a coin to decide whether to go through door #1 or door #2—when the time comes for making a move, the player must make one specific choice—e.g., go through *either* door #1 *or* door #2, but not both.[389]

The burgeoning field of "quantum game theory" relaxes the assumption of classical strategies to allow players to choose multiple discrete decision paths at once.[390] By allowing a player, for instance, to avoid choosing heads or tails on any given move—but, instead, a superposition of both possibilities, which is collapsed to a single move upon measurement at the end of the game—that player can enjoy strategic advantages relative to a more limited classical strategy space.[391] Such an enlarged "quantum strategy space" may yield radically different results when applied to traditional games.[392]

One limitation, however, of quantum game theory is its dependence on the players having access to a quantum computer or some other physical quantum system to implement an extended strategy space.[393] In this sense, standard quantum game theory relies on "exogenous" mechanisms to achieve its divergence from classical game theory.[394] Yet, by using post-classical power operators, legal actors can in effect choose multiple decision paths without relying upon the use of a quantum computer.[395] Recall that a quantum power operator may arbitrarily rotate a lower-order legal relation (typically, a first-order vector) in a complex Hohfeldian space.[396] In other words, a legal actor's implementation of a power need not result in classical legal entitlements—for example, on a

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[388] *See id.* at 4–5. *See also generally* David Meyer, *Quantum Strategies*, 82 PHYSICAL REV. LETTERS 1052, 1054 (1999) (introducing the notion of quantum strategies in game theory).

[389] *See* Sichelman, *supra* note 42, at 3 & n.1 (explaining the difference between a quantum strategy and a classical mixed strategy).

[390] *See id.* at 3–4.

[391] *See* Meyer, *supra* note 388, at 1052 ("Quantum strategies can be more successful than classical ones.").

[392] Hong Guo et al., *A Survey of Quantum Games*, 46 DECISION SUPPORT SYS. 318 (2008) (describing the results of a variety of quantum games).

[393] *See id.* at 320 ("[I]f classic games are played on a quantum computer or played by a quantum computer, the games can become quantum games.").

[394] *See* Sichelman, *supra* note 42, at 10 (positing the notions of exogenous and endogenous quantum games).

[395] *See id.* at 11 (describing how a legislature can engage in quantum-like strategies by enacting probabilistic laws that effectively result in superpositions of legal states).

[396] *See supra* notes 269–77 and accompanying text.

post-classical theory, a legislature may intentionally adopt a law that merely results in probabilistic entitlements.[397]

Arguably, these forms of legislation are the norm, not the exception, because legislatures almost by definition adopt laws that do not expressly provide for every situation that falls within the law's purview.[398] Such incompleteness in law-making is generally justified on two grounds: one, the large administrative costs in promulgating more specific laws;[399] and two, the benefits of flexible legal standards, which, for example, may allow adjudicators discretion in applying the law to unforeseen sets of facts before them.[400] However, the indeterminacy of incomplete laws is generally viewed by commentators and judges as a social cost.[401] On this view, if the legislature could prospectively imagine every potentially relevant factual situation and adopt a specific law for each one, society as a whole would be better off.[402]

The strategic benefits of indeterminacy present in "exogenous" quantum game theory's models indicate that there may be similar, yet "endogenous," benefits in legal games.[403] Elsewhere, I show that in the context of intellectual property (IP) games, indeterminacy can indeed have such an effect.[404] In particular, I compare the social welfare outcomes for classical (here, ironclad and deterministic) and quantum (here, fuzzy and probabilistic) IP rights, concluding that the latter may—under a range of reasonable assumptions—increase welfare in ways that are not easily

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[397] *See* Sichelman, *supra* note 42, at 11 (considering the legislature as a "mechanism designer" in the broad sense that can create classical or quantum-like rules).

[398] *See generally* Kenneth A. Shepsle, *Congress is a "They," not an "It": Legislative Intent as Oxymoron*, 12 INT'L REV. L. & ECON. 239, 250–51 (1992) ("it is best to think of statutes in terms similar to those in which incomplete contracts are treated in the economic theory of contracts"); Gillian K. Hadfield, *Incomplete Contracts and Statutes*, 12 INT'L REV. L. & ECON. 257, 257 (1992) (positively remarking on Shepsle's proposal).

[399] *See* Shepsle, *supra* note 398, at 252 ("Statute incompleteness and contract incompleteness are the result of the costs of legislating and contracting, respectively.").

[400] *See* D. Gordon Smith & Jordan C. Lee, *Fiduciary Discretion*, 75 OHIO ST. L.J. 609, 609–10 (2014) ("Real-world contracts are incomplete giving one or more of the parties some discretion over performance.").

[401] *See* Ehud Kamar, *A Regulatory Competition Theory of Indeterminacy in Corporate Law*, 98 COLUM. L. REV. 1908, 1919 (1998) (contending that "indeterminacy imposes high costs on individuals who try to plan their behavior so that it will meet legal requirements"); Grayned v. City of Rockford, 408 U.S. 104, 108–09 (1972) (remarking that "[v]ague laws offend several important values").

[402] *See generally* LON L. FULLER, THE MORALITY OF LAW 45 (rev. ed. 1969) (emphasizing the importance of predictable laws).

[403] *See* Sichelman, *supra* note 42, at 10–11, 14–15 (discussing "endogenous" quantum-like effects in the law).

[404] *See id.* at 15–21 (showing the probabilistic rights may result in benefits in patent race games that cannot be replicated with classical rights).

replicable by altering the scope or duration of classical IP rights.[405] Importantly, these benefits do not stem from savings in administrative costs or flexibility in application, but instead, in providing the government—as a mechanism designer of the rules of the IP game—strategic advantages not available from an ordinary, classical strategy space.[406] Specifically, by promulgating laws with no certain outcome, the government can provide incentives for ordinary legal actors to coordinate their behavior so as to act in ways that radically differ from those in the presence of determinate laws.[407] This "endogenous" quantum game theoretic approach extends the results obtained by scholars using neoclassical economic theories of behavior in the presence of incomplete information,[408] and provides a useful conceptual and mathematical framework for generating similar results in other areas of the law (e.g., property, contract, and public law).[409]

VI. APPLYING THE MATHEMATICAL FORMALISM OF SOCIAL LAW TO SCIENTIFIC LAW

How can "man-made" legal rules be described, at least in a "structural" sense, by the same mathematical formalism that governs the behavior of fundamental physical objects such as electrons, photons, and quarks?[410] In formal terms, because a spin-1/2 particle exists in a probabilistic superposition of one of two specific states (up or down) and collapses into one state upon measurement, the mathematical formalism describing the spin of a such a particle provides a starting point for formalizing the Hohfeldian legal relations.[411] Similarly, in the classical case, because a first-order legal relation exists in one state or the other (like a classical bit of information) and second-order powers merely flip these bits from one state to another, the mathematical formalism is quite similar

[405] *See id.* at 15–33 (examining a variety of models of probabilistic patent rights).

[406] *See id.* at 38 (noting that the government can act "as a mechanism designer us[ing] quantum-like strategies" to improve overall social welfare).

[407] *See id.* at 19–22 (describing the "anti-coordination" benefits of probabilistic patent rights).

[408] *See* Ayres & Klemperer, *supra* note 242, at 1015–16 (showing under a non-game theoretic, neo-classical model that probabilistic rights may lead to increased social welfare).

[409] *See* Sichelman, *supra* note 42, at 38–39 (stating that a contract can described as "an instrument to create right- and duty-qubits of specified probabilities[,]" not just "merely as an instrument that creates classical rights and duties among its parties").

[410] *See supra* notes 1–6 and accompanying text (discussing the relationship between scientific and social laws).

[411] *See supra* Part III.B.1 (positing that quantum spin is an analog to the post-classical, probabilistic Hohfeldian relations).

to analogous situations in the physical world of computational processes.[412]

Yet, is there a potentially deeper relationship between legal relations and physical relations and other rule-based systems, such as systems governed by scientific law?[413] One might answer this question affirmatively, positing that all rule-based systems must adhere *structurally* to the Hohfeldian formalism described earlier.[414] (Unfortunately, there is not sufficient space in what remains of this Article to assess this proposition in any rigorous sense. However, a brief discussion should provide a starting point for believing as much.[415])

There are three key reasons that support this proposition. First, like legal systems, all rule-based systems specify particular rules (often called "laws") that specify the behavior (more generally, states of affairs) that may or may not occur, either in a deterministic or probabilistic sense.[416] Of course, in some systems—like the legal system—it is understood that rules may be violated, but doing so typically entails some negative consequence.[417] Second, these rules concern specific actors (more generally, "subjects"), be it animate (persons) or inanimate (particles) that have a duty (or not) to obey first-order rules or are owed that duty (or

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[412] *See supra* note 178 (describing how the matrix-based operations of the Hohfeldian powers are reflected in the concept of computational, logic gates).

[413] *See* BLACKSTONE, *supra* note 4, at 38–44 (postulating that a deeper relationship exists between the "laws of nature" and "laws of nations" because they are both "rule[s] of action . . . prescribed by some superior, and which the inferior is bound to obey").

[414] *See supra* Parts II & III (proposing a logical and mathematical formalism of legal relations).

[415] *See infra* Parts VI.A & VI.B. (arguing that rules of games and physical laws adhere to the structural formalism of the Hohfeldian relations).

[416] *See generally* ANTONI LIGEZA, LOGICAL FOUNDATIONS FOR RULE-BASED SYSTEMS 91–93 (2006) (describing the general features of rules within the context of rule-based systems); Jaap Hage, *Building the World of Law*, 1 LEGISPRUDENCE 359, 363 (2007) ("Rules are for the institutionalized part of social reality what physical laws are for physical reality."). One may counter that physical "laws" do not share the same status as social laws, because physical "laws" as they are understood today stem from theories that in most (if not all) cases are known to be mere approximations to some "final" theory of reality. *See generally* STEVEN WEINBERG, DREAMS OF A FINAL THEORY: THE SCIENTIST'S SEARCH FOR THE ULTIMATE LAWS OF NATURE (2011). Yet, known social laws often occupy a similar epistemological status. Ultimately, law on the books—even if it is case law—is not "law in action," because law as it governs human behavior depends on enforcement patterns, which are often difficult to predict. *See* Leandra Lederman & Ted Sichelman, *Enforcement as Substance in Tax Compliance*, 70 WASH. & LEE L. REV. 1679 (2013) (explaining how lack of enforcement in effect changes the substance of the underlying law). Given this lack of knowledge in both areas, known "laws" will always be approximations to some "ground truth" that governs the behavior of the subjects of the law. *See id.*

[417] *See* Leonard Kreynin, *Breach of Contract as a Due Process Violation: Can the Constitution be a Font of Contract Law?*, 90 COLUM. L. REV. 1098, 1114 (1990) (remarking upon "the exceedingly rare case in which the state explicitly provides a right without a remedy").

not).[418] Third, the rules create a first-order duty (or not) with respect to some action–more generally, some state of affairs (i.e., the so-called object of the duty), which typically concerns the subjects of the duty.[419]

These three aspects of rule-based systems can be understood more concretely with examples. Here, I focus on two well-known rule-based systems: the game of chess and the classical and quantum mechanical motion of an electron in an electromagnetic field.[420] In so doing, I propose a novel approach using the Hohfeldian formalism to describe quantum measurement.[421]

A. Chess & Classical Rule Systems

Chess is a simple rule-based system in which the rules (in essence, laws) are fully determinate (i.e., classical). Chess rules are numerous.[422] Here, I choose one basic rule to illustrate how it adheres to the classical Hohfeldian formalism.[423] Specifically, if a player desires to move the rook

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[418] *See* BLACKSTONE, *supra* note 4, at 39–51 (contending that laws are "rule[s] of action" that must be obeyed by "those creatures" that have no free will and should be obeyed by those that do).

[419] For physical law, of course, this raises the question: To whom do the "objects" of physics (e.g., fundamental particles) owe their "obligations"? To the extent that physical law cannot be violated, the "subject" of the laws is not as relevant as for social law – in which the subject can typically bring suit for violation of the law. *See* Kreynin *supra* note 417 and accompanying text. Nonetheless, one fruitful approach is to view the subject to whom a physical "duty" is "owed" as the "universe-at-large," or "nature" more generally. *Cf. infra* notes 467–72 (discussing possible "subjects" of physical law). Importantly, such an approach inverts the usual view that deontic logic is a subset of modal logic – namely, that the logic of obligation is more particular than the logic of necessity. *Cf.* ERICH H. LOEWY, SUFFERING AND THE BENEFICENT COMMUNITY BEYOND LIBERTARIANISM 54 (1991) ("Generally, when an object follows a strict physical law, we do not use the language of obligation: We do not say that an egg dropped to the floor is compelled to break but rather that it is bound to do so."). However, the approach proposed here – consistent with that of Blackstone – conceives of modal logic as merely a subset of deontic logic in which the subjects have no will to violate the law. *See infra* notes 463–67 and accompanying text. Ultimately, one's view will depend upon whether one views physical law as "descriptive" or "prescriptive" in nature. *See id.* Nonetheless, regardless of one's view, the Hohfeldian typology, particularly second- and higher-order relations, can apply to a descriptive model in which certain processes can change underlying (descriptive) laws.

[420] *See infra* Parts VI.A & VI.B.

[421] *See infra* Part VI.B.

[422] *See, e.g.*, *FIDE Laws of Chess Taking Effect from 1 January 2023*, INT'L CHESS FED'N, https://handbook.fide.com/chapter/E012023 (last visited Sept. 6, 2024). Here, I only consider those rules that relate to how players can move the chess pieces on the board and not indeterminate rules, such as what counts as impermissibly distracting another player. *See id.* at r. 11.5 ("It is forbidden to distract or annoy the opponent in any manner whatsoever. This includes unreasonable claims, unreasonable offers of a draw or the introduction of a source of noise into the playing area.").

[423] *See id.* at r. 3.3 ("The rook may move to any square along the file or the rank on which it stands.").

(also known as a "castle"), the player may move it any number of squares vertically (forwards or backwards) or horizontally (left or right) to either (i) any unoccupied square, or (ii) any square occupied by an opponent's piece (which constitutes a capture); provided, however, that the rook cannot move through any other piece (whether her own or the opponent's piece).[424] (See Fig. 14.)[425]

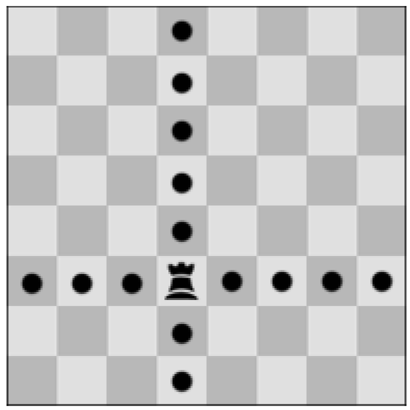

Fig. 14. Allowable Moves for a Rook in Chess

These two sub-rules related to moving a rook are "general" in the sense that they apply to the movement of any rook located on any square on the board for any configuration of other pieces.[426] The two sub-rules may become "specific" for a particular player given a unique set of positions of the chess pieces on the board.[427] For instance, suppose the chess board looks as displayed in Figure 15.

[424] *See id.* at r. 3.1 ("It is not permitted to move a piece to a square occupied by a piece of the same colour.").

[425] The figure indicates the vertical or horizontal motion a rook may take but does not indicate the aspect of the rule related to not moving through other pieces or capturing an opponent's piece.

[426] *See supra* Fig. 13 and accompanying text (describing how to form specific legal relations from more general legal rules).

[427] *See id.*

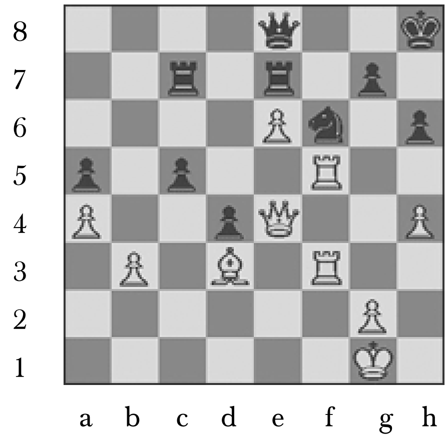

Fig. 15. An Arrangement of Chess Pieces in the Middle of a Game

If the player with white pieces is to move, and he desires to move his castle located at f5 in Figure 15, given the rules above, there are only seven possibilities,[428] which can be precisely specified by the first-order classical Hohfeldian formalism.[429] As a preliminary matter, notice that chess rule relations concern two actors, the two players of the game.[430] Because it is the player with white pieces' (hereinafter, "W") turn, he owes a Hohfeldian duty to the other player (hereinafter, "B") to move at least one of his pieces (since there is at least one legal move that can be made) or forfeit the game.[431] Subject to this duty, W is privileged in the Hohfeldian sense to move any piece W so desires, subject to the rules of the game.[432] If W chooses to move one of his rooks, then he is under a Hohfeldian duty to move the first rook to one of six squares or the second rook to one of seven squares—which one of the squares W so chooses is again W's Hohfeldian privilege.[433] All of these duties and privileges may be expressed by the exact same formalism as an ordinary classical legal rule—the two legal actors are B and W; there are duty and privilege relations (which are equivalent to strict-right and no-right relations) between B and W; and the

[428] If we label the horizontal axis from a to h and the vertical axis from 1 to 8, where the square in the lower left-hand corner is a1, then it is possible to precisely specify the "legal" moves of the castle located at f5. Specifically, the castle can move sideways, either to c5, d5, e5, g5, or h5; upwards to f6; or downwards to f4. *See FIDE Laws of Chess*, *supra* note 422, at C.1–13.

[429] *See id.* at art. 3 (using the deontic terms "permitted," "may," and "may not").

[430] *See id.* at r. 1.1 (noting that chess is between two players).

[431] *See id.* at art. 5 (stating the ways of winning, losing, and drawing a chess game).

[432] *See id.* at art. 3 (stating permissible moves).

[433] *See supra* note 423 and accompanying text.

states of affairs is W's movement of the pieces (here, the rooks) to a particular set of squares.[434] Formally, we can write the relation as:

> *J(W38) = BrW (1 (current state of the world). The chess pieces are on the squares as displayed in Figure 15; 2 (current state of the world). W has touched the castle located at f5; 3 (required future state of the world). W moves the castle either to c5, d5, e5, g5, h5, f4, or f6.)* (40)

This jural relation (which corresponds to the 38th move of *W* in the game, hence, *W38*) states that B has a *strict-right* vis-à-vis W that the state of affairs specified by statement (3) obtains given that the current state of the world is specified by statements (1) and (2).[435] That is, W is under a correlative duty that one of the states of the world specified by statement (3) obtains.[436] If one of these states of affairs in (3) do not obtain, then W violates his duty and loses the game.[437]

It should be obvious that given any specific arrangement of pieces on the board, the Hohfeldian formalism may be used to specify all of the duties and privileges of the player who must move (as well as, of course, all of the strict-rights and no-rights of the opposing player).[438] The first-order relations are all classical in the sense that there is no debate over whether there is a duty (or not) or privilege (or not)—that is, the arrangement of the pieces precisely and uniquely define all of the first-order relations between the players.[439]

What about the higher-order relations?[440] Because the rules of the game are fixed (at least with respect to how the players may move the pieces and win or lose the game), neither player has any second-order or higher-order powers, other than a "creation" power (starting the game),

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[434] *See supra* note 429 and accompanying text.

[435] Such a formulation aggregates different specific right-duty relations. In a more precise formulation, each of the states of affairs entails a separate right-duty relation. *See supra* Part II.A (describing the legal relations with respect to a specific state of affairs).

[436] *See id.* (noting that a Hohfeldian duty implies that unless a specific state of affairs obtains, there is a corresponding breach).

[437] *See FIDE Laws of Chess*, *supra* note 422, at art. 5 (stating how games are won, lost, and drawn).

[438] In other words, the synthetic process of applying the rules of chess to any specific arrangement of the pieces completely defines all the permissible and impermissible moves (i.e., ensuing states of the world that may or may not obtain) for any given arrangement of pieces on the board.

[439] *See supra* Part II.A (describing the classical legal relations).

[440] *See supra* Part II.B (discussing higher-order legal relations).

which brings legal relations into being, and an "annihilation" power (ending the game via a win or a forfeit) that destroys all legal relations.[441]

It is commonly thought that many, if not most, rule-based systems do not admit of second-order or higher-order powers being exercised by ordinary legal actors.[442] For instance, baseball players are not typically viewed as having the power to change the rules of the game, and cannonballs—while quite destructive—are not understood to have any power to change the laws of physics.[443]

In sum, the "laws" of chess, like the law of trespass, can be formalized—again, in a structural sense—by the classical Hohfeldian relations, using the same formalism as that presented earlier.[444] This is because chess laws, like trespass laws, are designed to constrain the behavior of legal actors (here, two chess players) through the specification of duties and privileges that may be inferred from the rules of the game, and that apply to specific states of affairs during the course of play.[445] Unlike the traditional legal system, however, the chess players (other than beginning and ending the game) have no second-order or higher-order powers—that is, they cannot change the rules of the game.[446]

B. Electron Motion & Quantized Rule Systems

Unlike the rules governing the movement of chess pieces, the rules governing certain systems are probabilistic, either in principle or practice.[447] One example of this type of system—or at least our best description of such a system—is the quantum field theory of electromagnetic fields.[448] The probabilistic nature of quantum field theory

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[441] *See supra* note 158 (describing creation and annihilation powers).

[442] *See* K.G. BINMORE, GAME THEORY AND THE SOCIAL CONTRACT: PLAYING FAIR 118 (1994) (explaining that in the context of game theory as well as scientific law, the rules of the game are assumed to be fixed).

[443] *Cf.* GRAHAM MCFEE, ON SPORT AND THE PHILOSOPHY OF SPORT: A WITTGENSTEINIAN APPROACH 178–85 (2015) (examining whether umpires can effectively change the rules of the game).

[444] *See supra* Part V.A (contrasting the "structure" and the "content" of the law).

[445] *See supra* note 3 and accompanying text (explaining how social law governs human action).

[446] As noted earlier, *see supra* note 422, by "rules" here, I do not refer to player conduct (e.g., distracting the other player), chess clocks, and other official rules not related to the movement of the pieces *per se* and winning, losing, or drawing the game.

[447] *See, e.g.*, MICHAEL STREVENS, BIGGER THAN CHAOS: UNDERSTANDING COMPLEXITY THROUGH PROBABILITY 9 (2006) (explaining how probabilistic behavior is a key aspect of complex systems).

[448] *See* JONATHAN DIMOCK, QUANTUM MECHANICS AND QUANTUM FIELD THEORY: A MATHEMATICAL PRIMER 161–69 (2011) (noting the probabilistic nature of quantum mechanics and quantum field theory).

is perhaps most apparent in Richard Feynman's "sum over histories" approach to describing the motion of a particle either in empty space or an external field.[449]

i. Electron Motion in Classical Electromagnetic Theory

Suppose we wish to describe the motion of an electron in an external electromagnetic field—assuming all other types of fields are of no consequence.[450] The Newtonian-Maxwellian approach to describing this motion is as follows.[451] First, we determine the initial position, x, and velocity, v, of the particle (i.e., at a time, t_0)—for instance, through a (classical) measurement—and label these values (assuming a Cartesian representation) x_0, y_0, z_0, and v_{x0}, v_{y0}, and v_{z0}, respectively.[452] Second, we determine the instantaneous acceleration of the particle at t_0 through Newton's second law, F = ma.[453] Here, F is the Lorentz force of the electromagnetic field, m is the mass of the electron, and a is the instantaneous acceleration of the electron.[454] We can describe F in terms of the electric field, E, and the magnetic field, B.[455] In this regard:

$$F = q(E + v \times B) \tag{41}$$

Furthermore, we can determine E (electric field) and B (magnetic field) from Maxwell's equations, which specify E and B in terms of each other and charge and current densities of electromagnetic field sources.[456] Third, once we know F, we can perform an integration to solve the position and velocity of the electron as a function of time, which is uniquely determined and fully deterministic.[457] In four-dimensional spacetime

[449] *See generally* RICHARD P. FEYNMAN & ALBERT R. HIBBS, QUANTUM MECHANICS AND PATH INTEGRALS ch. 2 (Daniel F. Styer ed., Dover Publ'ns 2010) (1965) (describing the "sum over paths" approach to quantum mechanics).

[450] *See generally* RAYMOND A. SERWAY & CHRIS VUILLE, COLLEGE PHYSICS chs. 15 & 19 (2014) (describing the motion of a particle in electric and electromagnetic fields).

[451] *See id.*

[452] *See id.*

[453] *See id.*

[454] *See id.*

[455] *See id.*

[456] I assume that we know that evolution of the charge and current densities over time as well. In reality, the path of the electron will affect these densities, which in turn will change the expected path of the electron. For simplicity, I ignore this interaction effect herein. *See, e.g.*, 3 JENÖ SÓLYOM, FUNDAMENTALS OF THE PHYSICS OF SOLIDS 63–65 (2010) (examining these types of interaction effects).

[457] *See* SERWAY & VUILLE, *supra* note 450, at chs. 15 & 19.

coordinates (i.e., three familiar spatial dimensions plus another dimension of time), the path of the electron will be a unique line starting at x_0, y_0, z_0, t_0, and ending at x_1, y_1, z_1, t_1.[458] Ignoring the y-z coordinates, one can draw a unique two-dimensional worldline for the particle from (x_0, t_0) to (x_1, t_1), such as that displayed in Figure 16.[459]

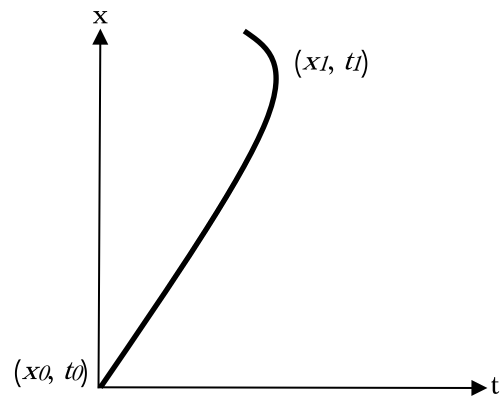

Fig. 16. The Classical Motion of a
Particle in Two-Dimensional Spacetime

How is all of this described in the Hohfeldian formalism? Essentially, given charge and current densities, an electron with an initial position and velocity, with no other possible influences, may be viewed as a legal actor of sorts, say P (for particle), that is under a Hohfeldian *duty* to follow the path specified by the procedure outlined above.[460] Even more so than the classical chess rules, the electron cannot possibly violate the duty.[461] That is, given the initial conditions of the electron, the charge densities, and the current densities, the electron is bound to follow the unique path described above.[462]

Typically, philosophers have used modal logic (the logic of necessity and possibility) rather than deontic logic (the logic of obligation and permissibility) to describe the effective "duties" of physical objects subject

458 *See* EDWIN F. TAYLOR & JOHN ARCHIBALD WHEELER, SPACETIME PHYSICS ch. 1 (2d ed. 1992) (explaining the notion of "spacetime").

459 *See id.*

460 *See supra* note 419 and accompanying text (addressing the "obligations" imposed by physical laws).

461 *See id.*

462 *See id.*

to physical laws.[463] In other words, because physical objects cannot in any manner violate physical laws—unlike social laws with human actors—the modal language of necessity and possibility is more apt to describe physical systems.[464] Yet, ultimately, modal logic can be conceived as homologous to deontic logic with the added assumption that the "actors" in modal logic (e.g., physical particles) cannot "violate the law."[465] As such, without much (if any) loss of generality, one can posit that particles, indeed all physical phenomena, are subject to deontic obligations—which, in turn, are identical to Hohfeldian duties.[466]

Nonetheless, it is hard to grapple with exactly to whom the electron owes its duty.[467] One possibility is to view the electron's duty as arising from the dictates of the local forces it experiences, particularly the local sum of those forces.[468] Extrapolating further, one might view the electron owing separate duties to the particular sources (i.e., particles) of the overall electromagnetic field that influences it, with the added principle that these duties may be "superimposed" upon one another, so that if the electron follows the additive, classical superposition of all these duties (i.e., forces),

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463 *See, e.g.*, Dalla Chiara, *supra* note 20, at 391; Deutsch & Marletto, *supra* note 21; Peter Mittelstaedt, *The Modal Logic of Quantum Logic*, 8 J. PHIL. LOGIC 479, 479 (1979).

464 *See, e.g.*, Hage, *supra* note 416, at 363 ("The law of gravitation exemplifies a *necessary* connection between states of affairs . . . which is a-temporal.") (emphasis added).

465 *Cf.* Mario Bunge, *Laws of Physical Law*, 29 AM. J. PHYS. 518, 526–27 (1961) (contemplating that meta-statements about physical law may be "deontic" but not recognizing as much for physical laws per se).

466 One might lodge John Stuart Mill's objection, as expressed by H.L.A. Hart, that natural laws "can be discovered by observation and reasoning [and] may be called 'descriptive' and it is for the scientist to discover them," whereas juristic laws "cannot be so established, for they are not statements or descriptions of facts, but are 'prescriptions' or demands that men shall behave in certain ways." HART, *supra* note 1, at 187 (commenting on JOHN STUART MILL, THREE ESSAYS ON RELIGION 3 (2d ed. 1874)); *see also* A.J. Ayer, *What is a Law of Nature?*, *in* THE CONCEPT OF A PERSON (1963) ("we do not conceive of the laws of nature as imperatives"). However, such a nominalist view wrongly assumes that describing scientific facts about the world somehow precludes the possibility of prescriptive, "natural" laws. Just as descriptive facts about human behavior (e.g., we observe that no vehicles travel faster than 45 mph on Main Street) do not preclude underlying prescriptive rules that guide such behavior (e.g., the speed limit on Main Street is 40 mph), descriptive facts about the "natural world" may be driven by underlying prescriptive rules. *See* E.J. Lowe, *Sortal Terms and Natural Laws: An Essay on the Ontological Status of the Laws of Nature*, 17 AM. PHIL. Q. 253 (1980) (adopting a normative account of physical laws). *Cf.* Christopher A. Fuchs, *Notwithstanding Bohr, the Reasons for Qbism*, 15 MIND & MATTER 245 (2017) (recasting quantum mechanics as a set of "normative aims in aid of [the decisions" of an individual observer). *See generally* Stephen Mumford, *Normative and Natural Laws*, 75 PHIL. 265 (2000) (examining whether natural laws should be viewed as "prescriptive" in the sense of social laws).

467 *See supra* note 419 (raising this question).

468 *See, e.g.*, HAFEZ A. RADI & JOHN O. RASMUSSEN, PRINCIPLES OF PHYSICS FOR SCIENTISTS AND ENGINEERS 184–86 (2012) (describing the net force as a linear sum of all acting forces).

it follows each duty owed to each particle creating the fields.[469] Another possibility is to view the electron as owing its duty to the "universe-at-large," U, which is analogous to owing a legal duty directly to the State.[470] Deciding which approach should be adopted is unnecessary here.[471] Rather, it is only important to recognize that the motion of an electron in a non-relativistic, classical field can be described structurally by the Hohfeldian formalism, just like the movement of a chess piece.[472] Namely, using U as the rights-holder:

> $J_1(t_0\text{-}t_1)$ = $U_{r1}P$ (1 (current state of the world). P is an electron that is initially situated at x_0, y_0, z_0 and with an initial velocity v_{x0}, v_{y0}, v_{z0}, all at time, t_0; 2 (current state of the world). There is a charge density and current density, ρ and J, which generate electric (E) and magnetic (B) fields; 3 (simplifying assumptions about the current state of the world). Only E and B affect the motion of electron, P, and the motion of P does not affect E and B; 4 (future state of the world). P follows a unique path in spacetime determined by conditions 1-3 and classical "laws" of motion determined by Maxwell's equations and Newton's laws) (42)

Here, $J_1(t_0\text{-}t_1)$ is a first-order legal relation that applies from the time t_0 to the time t_1.[473] The two legal actors are U, the "universe-at-large," and P, the electron.[474] U holds a first-order Hohfeldian *strict-right* vis-à-vis P that the state of affairs in statement 4 occurs.[475] More specifically, given existing

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[469] *See* ZEV BECHLER, NEWTON'S PHYSICS AND THE CONCEPTUAL STRUCTURE OF THE SCIENTIFIC REVOLUTION 265 (2012) (explaining that in the vision of Isaac Newton, "forces emanate from each particle towards every other particle in the infinite space, be they distance from each other as they may").

[470] *See* RICHARD FEYNMAN, THE CHARACTER OF PHYSICAL LAW 54 (1965) (remarking that "[a]t present we believe that the laws of physics have to have [a] local character . . . but we do not really know").

[471] *Cf.* NANCY CARTWRIGHT, HOW THE LAWS OF PHYSICS LIE 49 (1983) ("God may have written just a few laws and grown tired. We do not know whether we are in a tidy universe or an untidy one. Whichever universe we are in, the ordinary commonplace activity of giving explanations ought to make sense.").

[472] *See supra* Part VI.A (applying the Hohfeldian formalism to the rules of chess).

[473] *See supra* Part II.A (describing the first-order legal relations).

[474] *See id.* (discussing the role of legal actors relative to first-order legal relations).

[475] *See supra* Part II.A (discussing how if A holds a strict-right vis-à-vis B that some state of affairs obtains, then B is under a corresponding duty to A that state of affairs obtains).

conditions (1 and 2) and simplifying assumptions (3) regarding the current state of the world, P is obligated to perform (4).[476]

Note that Hohfeldian relation in (42) is "structural" in the sense that it only describes *P*s duty to follow a certain path given particular initial conditions and assumptions, but it does not specify *how or why* the initial conditions and assumptions cause *P* to follow such a path—such a description is the purview of electromagnetic theory and not Hohfeldian structural analysis.[477] Moreover, like the game of chess, the Hohfeldian relations in this example are fully first-order: nothing within the classical system can change the laws that govern motion of the subjects (particles).[478] Changes (and the origins) of the laws are considered "metaphysical" or even "religious" concerns.[479]

ii. Electron Motion in Quantum Theory

In quantum theory, instead of traveling along one unique path from (x_0, t_0) to (x_1, t_1) as in the classical case, a particle whose wave function has not yet been measured may be viewed as effectively traveling along *every possible path* from (x_0, t_0) to (x_1, t_1), and where unlike classical electrodynamics, there may be a probability for (x_1, t_1) to be one of many,

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[476] *See id.* (noting that states of affairs often comprise specific actions that the dutyholder must perform (or not)).

[477] Related, any mathematical formalization of (42), e.g., through the vectors and matrix formalism described earlier, concerns only *P*s *duty per se* to follow a certain path through spacetime, but does not provide any information about the mathematics concerning the path itself. Similar to the earlier distinction between legal structure and legal content, in the context of scientific laws, "structure" defines the underlying nature of law, while the merger of "content" into that structure specifies the precise mathematical form of states of the world that must obtain (or not). *Cf. supra* Part V.A (describing how legal "content" must fill legal "structure" via a "synthetic" process to generate specific legal propositions).

[478] To be certain, a handful scholars have recently contended that the laws of physics may change over time, but the overriding view among physicists is the traditional, classical one that the "ultimate" physical laws are timeless. *See* ROBERTO MANGABEIRA UNGER & LEE SMOLIN, THE SINGULAR UNIVERSITY AND THE REALITY OF TIME: A PROPOSAL IN NATURAL PHILOSOPHY 259–92, 447–76 (2014). Interestingly, one of these scholars, Roberto Unger, is a law professor and leading Critical Legal Studies proponent. Unger, *supra* note 3131. Nonetheless, under Unger and Smolin's proposal that physical law may change over time, such changes are endogenous to the (first-order) physical world, and they do not reference, much less incorporate, analytic legal theory. *See* UNGER & SMOLIN, *supra* note 478, at 259–92, 447–76.

[479] *See* Edgar Zilsel, *The Genesis of the Concept of Physical Law*, 51 PHIL. REV. 245 (discussing historical approaches to the "concept of physical law," all grounded in religion or metaphysics); *see also* TIM MAUDLIN, THE METAPHYSICS WITHIN PHYSICS (2007) (addressing the nature of physical law from a philosophical, metaphysical perspective). *Cf.* RONALD N. GIERE, SCIENCE WITHOUT LAWS ch. 5 (1999) (expressing skepticism "regarding the role of supposed laws of nature in science").

often an infinite number, of endpoints.[480] In Figure 17 below, four of those paths to a particular (x_1, t_1) are displayed—of course, it is impossible to show every path to (x_1, t_1), much less all of the possible (x_1, t_1) endpoints for which there is a non-zero probability of measurement.[481]

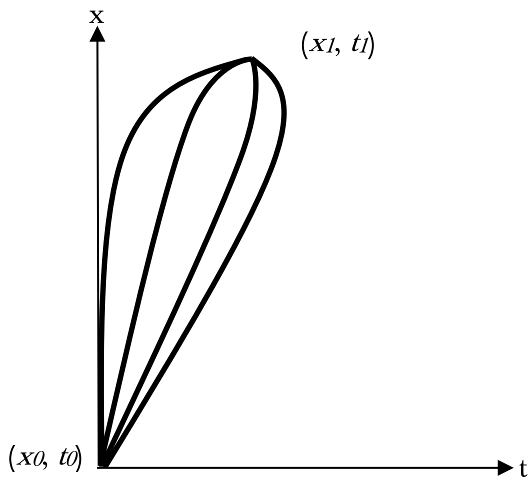

Fig. 17. The Quantum Motion of a
Particle in Two-Dimensional Spacetime

If we let (x_1, t_1) be any general point in spacetime (x, t), the Feynman "sum over histories" approach provides a method of calculating a sum of weighted "contributions" from each path to the evolution in spacetime of a quantum state $|\psi\rangle$ of a particle P to a given (x_1, t_1).[482]

Such an approach in many ways resembles decision theory, in which an actor—for instance, a legal actor—chooses a decision path at various nodes in time in a branching network based on the actor's estimates of how a given law applies to those actions (see Fig. 18).[483] The total

[480] *See* R.P. FEYNMAN & A.R. HIBBS, QUANTUM MECHANICS AND PATH INTEGRALS 28–39 (1965) (describing the basic approach to using multiple paths and associated path integrals in quantum mechanics).

[481] *See id.* at 30 (showing a similar diagram).

[482] *See id.* at 32–38. Ultimately, quantum field theory treats particles as the emanations of underlying fields, but for simplicity, I refer to the particle in the discussion that follows. For the conceptual difficulties involved in classifying "particles" as basic objects in quantum field theory, *see* Meinard Kuhlmann, *Quantum Field Theory*, *in* STANFORD ENCYCLOPEDIA OF PHILOSOPHY (Edward N. Zalta & Uri Nodelman eds., Summer 2023), https://plato.stanford.edu/archives/sum2023/entries/quantum-field-theory/.

[483] *Decision Tree – The Graphical Representation of Decision Options*, T2INFORMATIK, https://t2informatik.de/en/smartpedia/decision-tree/ (last visited Sept. 9, 2024) (presenting and explaining a typical decision tree); *see also* Lars Lindahl, *Hohfeld Relations and* Spielraum *for*

probability that an actor arrives at a specific node is the weighted sum of the probabilities that an actor takes a particular path through the tree that may end at that node.[484]

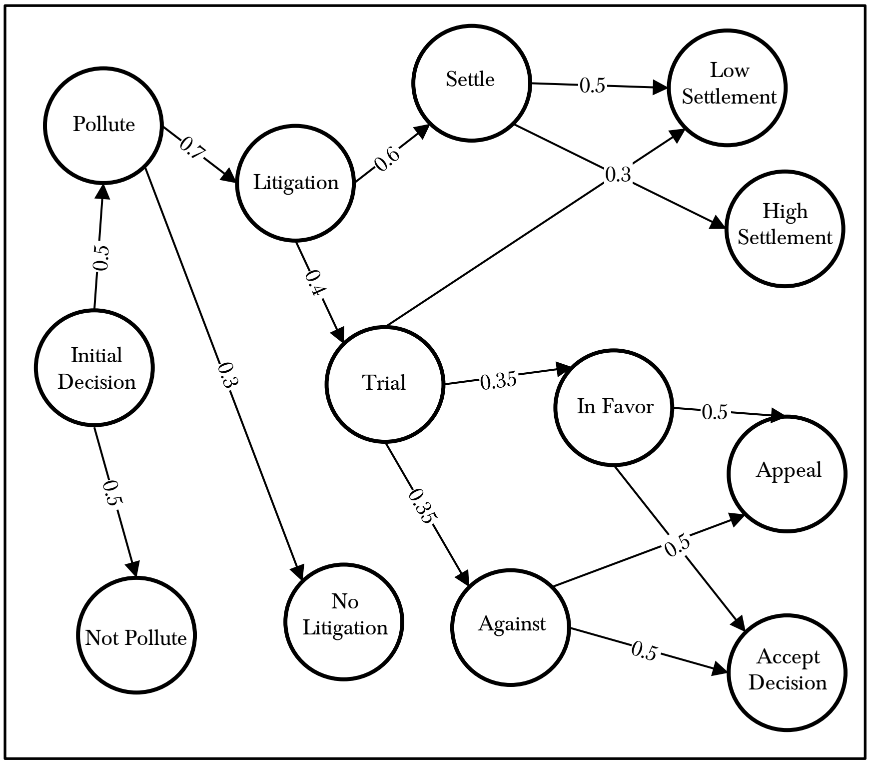

Fig. 18. Example of a Decision Tree with Multiple Paths to End Nodes[485]

In classical decision theory, the paths to various nodes are non-interacting, and the probabilities for individual paths to a given node may be simply summed to achieve a total probability to reach that node.[486] However, in quantum field theory, the paths "interfere" with another so

Action, 26 ANÁLISIS FILOSÓFICO (2006), http://ref.scielo.org/vgpn58 (relating the Hohfeldian scheme to decision theory).

[484] *See* Michael I. Jordan et al., *Hidden Markov Decision Trees*, *in* 9 ADVANCES IN NEURAL INFORMATION PROCESSING SYSTEMS 501, 502–03 (Michael C. Mozer et al., eds., 1997) ("The total probability of an output given an input is the sum over all paths in the tree from the input to the output.").

[485] *See* ChatGPT, Decision Tree in Legal Game Theory Scenario, OPENAI (last visited Nov. 26, 2024) (using graphic created via repeated prompting).

[486] *See* Jordan et al., *supra* note 484, at 502–03 (describing how to determine probabilities to particular nodes in a decision tree).

that their contributions result in a complex calculation that determine a probability for a particle reaching a particular endpoint, (x_1, t_1), but no clear probability that a particle took any particular path.[487]

What would the Hohfeldian relations be in this situation? As an initial matter, note that in the absence of measurement— that is, in the absence of any the attempt to determine where the particle is located —the evolution of a quantum state, $|\psi\rangle$, of a particle, P, is *from a Hohfeldian standpoint*, entirely *classical*.[488] Specifically, in quantum theory, one can derive a *deterministic* unitary evolution operator, $U(x, t)$ (not to be confused with U, the universe-at-large), which specifies how a particle's quantum state, $|\psi\rangle$, evolves in spacetime.[489]

In this sense, as long as no measurements are made, a particle in effect deterministically "travels" along every path possible—not in any physical sense, but in a quantitative sense—and its doing so may be described by a classical Hohfeldian relation.[490] In particular:

[487] *Cf.* David Deutsch, *Quantum Theory of Probability and Decisions*, 455 PROC. ROYAL SOC'Y LONDON A 3129 (1999) (claiming to derive the probabilistic axioms of quantum mechanics from the non-probabilistic axioms of quantum mechanics and the non-probabilistic portion of decision theory).

[488] *See supra* Part II.A (presenting a formal, logical model of the classical Hohfeldian relations). To be certain, this does not mean the evolution of a quantum state is described by classical physics.

[489] *See* Stephen L. Adler, *Why Decoherence Has Not Solved the Measurement Problem: A Response to P.W. Anderson*, 34 STUD. HIST. PHIL. MOD. PHYSICS 135, 137–39 (2003) (describing the unitary evolution of a quantum system prior to measurement as deterministic).

[490] *See* FEYNMAN & HIBBS, *supra* note 480, at 237–56 (describing the fundamentals of quantum electrodynamics); HENRYK AROD & LESZEK HADASZ, LECTURES ON CLASSICAL AND QUANTUM THEORY OF FIELDS 253 (2017) ("The time evolution of the states of an isolated quantum system is described by a unitary operator U in Hilbert space. Path integrals are used in order to write matrix elements of U . . . ."). For simplicity, I ignore the effects of the field on the electron's quantum spin. Also note that while the previous discussion of quantum Hohfeldian relations concerned the spin of a particle, a discrete quantity, here the concern is the ultimate position of the particle, which is a continuous quantity. As explained below, for simplicity, I assume the particle can only reach two possible endpoints. However, the discussion here can be suitably extended to a continuum of possible endpoints.

$J_1(t_0\text{-}t_1) = U_{T1}P$ (1 (current state of the world). P is an electron, associated with a quantum state $|\psi\rangle$, which is in an initial state $|\psi_0\rangle$ at an initial point in spacetime (x_0, t_0); 2 (current state of the world). There is a known charge density and current density, ρ and J, that generate an external electromagnetic field;[491] 3 (simplifying assumptions). The only field that affects the evolution of $|\psi\rangle$ is the electromagnetic one generated in 2. and the motion of P does not affect ρ and J; 4 (future state of the world). $|\psi\rangle$ must uniquely evolve according to a unitary operator $U(x, t)$ that is derived according to the Feynman "sum over histories" (or equivalent) approach, which takes into account a contribution from every possible path from (x_0, t_0) to a given (x_1, t_1); 5 (future state of the world) The evolution $|\psi\rangle$ with respect to all possible endpoints, x_i, after some amount of time t_1 is determined by calculating the probability of going from (x_0, t_0) to (x_i, t_1) according to the "sum over histories" approach for every possible x_i. (43)

Upon a quantum measurement, however, the Hohfeldian formalism no longer remains classical, and must be described by the quantum Hohfeldian formalism discussed earlier.[492] So far the discussion has focused on the probability of measuring a particle at a single endpoint. In general, there are multiple possible endpoints, and one calculates the probability of measuring the particle at each of the possible endpoints. Simplifying further, suppose that there are only two possible endpoints, (x_1, t_1), at which the particle may terminate, and only two paths to each individual endpoint.[493] In this case, a position measurement designed to determine which of the two endpoints the particle P arrived at will in effect collapse the wave function $|\psi\rangle$ in such a way that the particle P associated

[491] For simplicity, it is assumed that the electromagnetic fields are not quantized. *See generally* WALTER E. THIRRING, PRINCIPLES OF QUANTUM ELECTRODYNAMICS 53–54 (Academic Press 2013) (1958) (noting that in quantum electrodynamics the electromagnetic fields are quantized).

[492] *See generally* SERGIO ALBEVERIO ET AL., MATHEMATICAL THEORY OF FEYNMAN PATH INTEGRALS: AN INTRODUCTION 133 (2d ed. 2008) ("[T]he state of the system after the measurement is the result of a random and discontinuous change, the so-called 'collapse of the wave function', which cannot be described by the ordinary Schrödinger equation."); M.B. MENSKY, CONTINUOUS QUANTUM MEASUREMENTS AND PATH INTEGRALS (1993) (proposing a model of quantum measurement in the context of Feynman path integrals).

[493] Knowledgeable readers will recognize that this description is highly simplified—e.g., it assumes a finite-state, as opposed to a continuous, infinite-state system. However, a more faithful description would not alter the substance of the remarks made herein.

with the wave function will be, in our idealized example, found to be only at one endpoint or the other.[494] Yet, oddly, the position measurement as constructed cannot determine which of the two remaining possible paths the particle P traveled on to reach the specific endpoint—indeed, such a question may be ill-formed so as to have no answer.[495]

In our idealized example, we can attempt to determine the specific path a particle takes by placing measuring devices in the middle of each of the four paths, and when a position measurement is made in this instance, only one of the four paths will register.[496] Yet by measuring the wave function, $|\psi\rangle$, in order to "localize" the particle in a single path, this will "destroy" the interference effects that inform us of the probabilities of reaching the two destinations of interest.[497] In essence, the sum-over-histories approach—like the standard approaches of quantum mechanics—prevents acquiring full information regarding both the path traveled and the final destination of a particle's motion.[498]

Regardless, whether one measures the path or ultimate position, measurement—like a legal judgment—results in a classical measurement outcome and thereby eliminates possible historical states of evolution of the particle.[499] In some configurations, measurements will result in selecting a single classical destination point, albeit in many cases, a destination point that will deviate from the one otherwise dictated by classical law.[500] In other configurations, rather than selecting an endpoint, measurement can result in selecting a single, classical path in the Hohfeldian sense, albeit in many cases, a path that will deviate from the

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[494] *See generally* MENSKY, *supra* note 492, at 168–81 (discussing quantum collapse in the context of the path integral formulation of quantum mechanics).

[495] *See* MICHAEL G. RAYMER, QUANTUM PHYSICS: WHAT EVERYONE NEEDS TO KNOW 86–87 (2017) ("It is more accurate to say the photon does not actually exist at any particular place rather than to say it exists in two places at once.").

[496] *See generally* MENSKY, *supra* note 492, at 168–81 (describing a process of "continuous" quantum measurement that selects specific paths along a possible set of trajectories).

[497] As noted earlier, this description is somewhat simplified, because in some senses it reifies the notion of particle and wave, when the quantum mechanical formalism is more specifically a means to calculate probabilities of some "detector firing." Anton Zeilinger et al., *Happy Centenary, Photon*, 433 NATURE 230, 233 (2005) (casting doubt on the traditional view that measurement destroys interference effects).

[498] *See* L.S. SCHULMAN, TECHNIQUES AND APPLICATIONS OF PATH INTEGRATION 12–16 (2005) (highlighting the inherent limitations of the sum-over-histories approach in obtaining full information about a particle's path and final destination due to quantum uncertainty).

[499] *See* Wojciech H. Zurek, *Decoherence, Einselection, and the Quantum Origins of the Classical*, 75 REV. MOD. PHYSICS 715, 728–29 (2003) (emphasizing that quantum measurement produces classical outcomes, which in turn eliminates possible states of the particle's evolution).

[500] *See generally* ALBERT D. MESSIAH, QUANTUM MECHANICS 41 (2014) (discussing how quantum mechanics leads one to "renounce the classical equations of motion").

single path otherwise dictated by classical physical law.[501] In either sense, quantum measurement results in "classical-like" measured states, though ones subject to an "uncertainty principle" that prevents acquiring complete knowledge about a particle's complementary properties or histories, to the extent those concepts are meaningful.[502]

Because the structure of quantum Hohfeldian relations uses the very mathematics of quantum mechanics, in the event one measures a specific path the particle is traveling along, instead of viewing the particle as subject to *one classical legal relation* wherein the particle is obligated to move along a certain path with particular probabilities p_i "revealed" upon measurement, one may posit *four separate legal relations* that can be used to describe the four possible results (i.e., four particular paths) of quantum measurement.[503] Thus, like a legal actor who may in essence be subject to different laws depending on the result of a judicial ruling, the particle here is potentially subject to four separate first-order legal relations–i.e., four separate classical-like laws of nature.[504] In a Hohfeldian sense, it is as if each potential path of the particle from (x_0, t_0) to (x, t) represents a different classical-like law that may be instantiated for the particle P depending on the result of a measurement of the particle's particular path.[505] When a suitable measurement selects only one path, the selected path becomes a *positive duty* relation, and the unselected paths become *negative duty* relations.[506]

Suppose, for instance, measurement effectively selects path 3–*in essence, classical-like "law 3"*–as governing the behavior of the particle P as it travels from (x_0, t_0) to (x, t).[507] And from the earlier discussion, we

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501 *See generally* 3 RICHARD P. FEYNMAN, ROBERT B. LEIGHTON & MATTHEW SANDS, THE FEYNMAN LECTURES ON PHYSICS 1–6 (2011) (discussing how measurement may result in the selection of one classical path in the context of the double-slit experiment).

502 *See* WERNER HEISENBERG, THE PHYSICAL PRINCIPLES OF THE QUANTUM THEORY 23–26 (1949) (discussing the concept of quantum measurement in the context of the uncertainty principle).

503 *Cf. id.* at 79 (noting that when interference phenomena disappear, the path of the particle can be compared to the classical path).

504 These paths are "classical-like" in the sense that they dictate a single path through spacetime, but notably differ from the actual classical laws in that only one path is dictated by the laws of classical physics. *See* SHAMIL U. GALIEV, EVOLUTION OF EXTREME WAVES AND RESONANCES 467–69 (2020) (discussing alternative possible paths in the Feynman path integral formulation of quantum mechanics).

505 *See id.* (noting that the alternative paths are impossible in classical mechanics).

506 *See supra* note 117 and accompanying text (discussing positive and negative duties).

507 The fact that law 3 is selected does not necessarily mean the particle actually traveled along path 3 prior to measurement. Rather, it may simply be that the particle remained in a superposition of states until measurement, at which point law 3 is selected and thereafter the particle merely acts *as if* it had travelled along path 3 prior to measurement.

know—at least in the Hohfeldian sense—how laws are chosen: via higher-order powers.[508] Specifically, a second-order Hohfeldian power alters the relevant probabilities of each $|j(i)\rangle$ at (x_i, t_i) (i.e., the probability of each path being taken) so that only one path remains with a 100% probability.[509] Thus, like a judge who decides whether a plaintiff is subject to a given law or not via a second-order power, *a quantum measurement executes the analogue of a second-order Hohfeldian power to collapse the wave function $|\psi\rangle$ of a particle P.*[510] This collapse chooses one of many competing classical-like states—in effect, laws—the particle could have obeyed before the measurement.[511]

The realization that the collapse of the wave function is not a first-order physical process, such as a particle obeying a classical electromagnetic field, cures many of the conceptual difficulties and so-called paradoxes of quantum measurement.[512] First, it helps to resolve the paradox of how a measuring device—*that itself is described by quantum mechanics*—can collapse a quantum mechanical state of a particle that is coupled to the measuring device in a quantum mechanical fashion.[513] In particular, although a measuring device may be completely quantum mechanical in nature—it is so only in a *first-order sense*—that is, all of the components of the measuring device have first-order duties, just like the

[508] *See supra* notes 236–55 and accompanying text (discussing how a second-order power leads to judgment among multiple probabilistic first-order relations, resulting in a classical single first-order relation). In essence, the second-order measurement power in quantum mechanics will (nearly) instantaneously convert a probabilistic superposition of states, $|\psi\rangle$, into some determinate eigenstate of $|\psi\rangle$ that one might expect from a classical measurement (albeit a result that might not follow from the "macroscopic" classical law that is the "large-scale" limit of the quantum formalism). *See generally* RUTH E KASTNER, JASMINA JEKNIC-DUGIC & GEORGE JAROSZKIEWICZ, QUANTUM STRUCTURAL STUDIES: CLASSICAL EMERGENCE FROM THE QUANTUM LEVEL (2011).

[509] *See supra* notes 226–45 and accompanying text (noting that judgment eliminates the possibility of all but one lower-order legal relation).

[510] In essence, the only means by which a true (i.e., quantum-like) superposition of possible first-order states of any system can be reduced to a single, first-order state is via a second-order process. This approach is in contrast to the mere absence of knowledge of which specific first-order state a system occupies, which in turn is revealed by a classical measurement via a first-order physical process. It also differs from theories that explain measurement via first-order physical processes that transform the state of the quantum system.

[511] *See supra* notes 251–55 and accompanying (describing how in effect judgment selects a particular law that governs the legal actor at-issue).

[512] *See* NORSEN, *supra* note 236, at 148–49, 178–82 236(explaining the paradoxes and problems that arise from current approaches to quantum measurement).

[513] *See id.* at 59–69 (explaining how conceptual difficulties arise if one assumes that the measuring device is also a system wholly described by quantum mechanics).

particle it measures, to obey the laws of quantum mechanics.[514] And these first-order duties extend to the interaction of the measuring device's constituent particles with that of the particle being measured.[515] However, when the measuring device "makes" a measurement, it is exercising a *second-order power*—that is, the device is changing the nature of the first-order quantum states of the coupled measuring device-particle system via a separate second-order process.[516]

Just like a judge that is subject to first-order legal relations vis-à-vis particular litigants—e.g., a judge is under a first-order duty not to jump over the bench and punch the litigants—such restrictions do not affect the judge's ability to take a second-order action to decide a case. Similarly, a measuring device's second-order (i.e., measuring) *powers* are *not* described by first-order quantum mechanical theory.[517] To describe the second-order process of quantum measurement with a first-order language would be akin to describing the process of judicial decisionmaking via first-order *strict-rights* and *duties*.[518] Such a view, like the existing views of quantum

[514] *See id.* (noting that since a measuring device is a physical system, it is reasonable to assume that it is also described by the quantum mechanical formalism).

[515] In other words, when a quantum measuring device—more broadly, any physical system—interacts with another physical system, the first-order nature of this interaction is entirely described by quantum mechanics absent any measurement process.

[516] More generally, one might imagine it is not the measuring device per se exercising the second-order power, but rather that the coupling of the system to the measuring device effectuates a second-order process in the universe-at-large that "measures" the state of the system. *See generally* HEELAN, *supra* note 195, at 75 (discussing classical measurement in statistical mechanics).

[517] *See* NORSEN, *supra* note 236, at 148–49, 178–82236 (explaining the paradoxes and problems that arise from current approaches to quantum measurement); ROGER PENROSE, THE ROAD TO REALITY: A COMPLETE GUIDE TO THE LAWS OF THE UNIVERSE 782–84 (2005) (arguing that the measurement problem arises from the apparent conflict between the deterministic unitary evolution of the wave function and the collapse of the wave function during measurement and is not currently resolved within the standard formalism of quantum mechanics). In this regard, although quantum measurement is a non-unitary process from a first-order perspective—because the quantum system apparently "collapses" from a superposition to a single eigenstate of the system in a seemingly irreversible manner—it is unitary *from a second-order perspective*. In other words, like a legal judgment, there is a *second-order* rotation of the quantum state from a superposition of potential states to one specific eigenstate, preserving the norm of the quantum state in the sense that all non-measured states have coefficients of zero, such that the state becomes "classical-like" (in the Hohfeldian sense). It is only from a first-order perspective that the second-order rotation is instantaneous and irreversible, but from a second-order perspective it occurs in (second-order) time and is reversible (by second- and higher-order processes). The precise nature of quantum measurement will be explored in a later paper. Ted Sichelman, *Quantum Measurement as a Second-Order Physical Process* (working paper) (on file with author).

[518] *See generally* HART, *supra* note 11, at 79–91 (discussing the differences between rights and powers).

measurement, would immediately lead to conceptual problems.[519] Thus, the fact that a quantum measuring device is described by (first-order) quantum mechanics is of no issue to its (second-order) ability to collapse the wave function of particular particles. Notably, in so doing, the measurement device will also collapse the wave functions of its own constituent particles that are coupled to the particle being measured so as to create a "record" of the state of the measured particle.[520]

In this regard, it is important to distinguish changes brought about in a quantum system—and its associated state vector—by ordinary first-order physical processes and changes brought about by second-order physical processes.[521] Recall that for legal relations, a first-order legal relation can be changed by either (1) "ordinary" actions in the world that are *not* the result of volitional, intentional acts designed to change legal relations; or (2) volitional, "legal" acts carried out with the intention of changing legal relations.[522] For instance, ordinary actions may encompass changing ethical views in society about a particular behavior, which in turn affect a judge's decision in a close case, so as to rotate the legal relations vectors even though no legal power has been exercised.[523] In other situations, mere changes in the state of the world—e.g., whether the apple season has been unusually dry so as to result in a poor yield—may change a disability

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[519] *See id.* (noting the conceptual difficulties of describing powers in the language of rights and duties).

[520] *See* NORSEN, *supra* note 236, at 59–69 (noting that if the measuring device is coupled to the quantum system, the process of measurement must also collapse the wave function of the measuring device itself). To be certain, whatever specific "record" the measuring device creates will not generally dictate any future measurement of the system-of-interest as it continues to evolve. In general, as soon as a quantum state collapses, it will then evolve again as a quantum superposition of states. *See* DENNIS DIEKS & PIETER E. VERMAAS, THE MODAL INTERPRETATION OF QUANTUM MECHANICS 17 (2017).

[521] More specifically, first-order processes can be completely described by the unitary evolution of the wave function of a system or coupled systems, for instance, in non-relativistic quantum mechanics by the Schrodinger equation or the Feynman sum-over-histories approach, which in turn can be described by Hohfeldian formalism noted earlier. *See supra* note 490 and accompanying text. In contrast, second-order processes are described by a formalism that in effect alters the underlying dynamics of the first-order system. Currently, this alteration is inserted by hand in quantum mechanics by the non-unitary "collapse of the wave function" and all attempts to describe this process in more detail have in effect relied on a first-order formalism. Interestingly, the many worlds theory could be regarded as a second-order approach to the measurement problem, though one that has yet to be formally described in second-order language. However, as discussed below, the second-order approach proposed here dispenses with the need for the many-worlds approach, which presents other conceptual problems. *See infra* notes 536–50 and accompanying text.

[522] *See* Sichelman, *Annotated Fundamental Legal Conceptions*, *supra* note 14, at 46 n.52 (modifying Hohfeld's original approach to describing changes in legal relations).

[523] *See* RICHARD A. POSNER, HOW JUDGES THINK 96–101 (2008) (describing the various forces that shape social ideologies, which in turn influence judicial decisionmaking).

to a power, providing the option to a farmer to exercise rights under an insurance agreement, and so forth.[524] Contrast these "ordinary" actions that result in changes in legal relations with the volitional "legal" acts of a legislature, regulatory agency, or court, which intentionally effectuate changes in underlying legal relations.[525] Notably, the volitional act of a judge in deciding a close legal case—even including where the judge is "bound" to follow a statute, at least in the post-classical framework—sounds in the nature of a legal power, the same type of power that the legislature exercises when "making" law.[526]

In the context of quantum mechanics, the ordinary, non-volitional actions that change legal relations are analogous to first-order physical processes that change the physical states of a relevant quantum system.[527] For instance, a changing electric field may change the underlying state of quantum system, thereby changing the probabilities that the system will be measured in certain states.[528] The changing electric field is akin to the effects of an unusually dry apple season—namely, a first-order action—on the likelihood that an insurance policy is triggered.[529] Although the unusually dry weather and the changing electric field may alter first-order states, importantly, neither changes the underlying *rules* governing the system.

Such first-order physical processes are in contrast to second-order processes, which—at the most fundamental level—*alter, create, or terminate the laws themselves*.[530] Like the second-order, limited "lawmaking" power of a judge to "collapse" a legal system in a superposition of first-order states, quantum measurement is a limited "lawmaking" power that collapses a

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[524] *See generally* Shiva S. Makki & Agapi Somwaru, *Farmers' Participation in Crop Insurance Markets: Creating the Right Incentives*, 83 AM. J. AGRIC. ECON. 662 (2001).

[525] *See supra* note 145 and accompanying text (discussing the Hohfeldian approach to changes in legal relations).

[526] *Cf.* Einer R. Elhauge, *Does Interest Group Theory Justify More Intrusive Judicial Review?*, 101 YALE L.J. 31, 47 n.78 (1991) ("From an opposite perspective, one might object that narrowing statutory construction is not an expansion of judicial lawmaking authority because, given the ubiquitous ambiguity of statutes, anything other than narrow statutory construction involves judicial lawmaking.").

[527] *See, e.g.*, A. S. DAVYDOV, QUANTUM MECHANICS INTERNATIONAL SERIES IN NATURAL PHILOSOPHY 288–89 (describing the changing quantum state of an atom when subject to an external electric field).

[528] *See id.* (describing the changing energy states of the system).

[529] *See supra* note 524 and accompanying text.

[530] *See supra* Part III.B.1 (describing the quantum formalization of the Hohfeldian relations, including the user of quantum Hohfeldian powers to "collapse" a legal system in a superposition of first-order states).

physical system in a superposition of first-order states.[531] As with the comparison between the limited lawmaking power of a judge and the more general lawmaking power of a legislature, notably—barring constraints—a second-order physical process could theoretically change the law of gravity from an inverse-square to an inverse-cube rule.[532] Like the exercise of legal powers, such changes are not the result of ordinary changes in the state of the world.[533]

The mistake made by current approaches is to view quantum measurement as a first-order process—namely, as arising from ordinary changes in the states of the world.[534] Like views of adjudication that attempt to categorize the judge merely as an "umpire calling balls and strikes," rather than as exercising judicial power to change legal relations themselves, because all current interpretational approaches to quantum measurement fail to appreciate the second-order nature of that process, they in turn fail to capture the effective "power" (a *second-order* physical power) that ultimately "collapses" the quantum state.[535]

[531] *See supra* notes 498–515 (describing quantum measurement as a second-order physical process). Notably, external first-order processes that alter the state of the quantum system typically will not alter the state in such a manner as to eliminate all but one of the potential measurement states of the system, but rather will change the probabilities with respect to the entirety of the potential measurement states. Although in some unusual cases, one state may remain from the result of the first-order process, generally, multiple states will remain, underscoring the difference between a first-order and second-order physical process. One might attempt to rebut such a view by analogizing contingent legal relations to hidden variable theories, arguing that measurement is a first-order process that merely "reveals" the actual physical state of the quantum system. In other words, like a duty that is not yet active, but is triggered by some external event other than the exercise of a legal power, measurement might be viewed as a first-order physical process that "triggers" the underlying *and pre-existing* physical state of the system. However, such (first-order) hidden variables theories present a host of conceptual difficulties. Moreover, there is no hidden variable theory that explains precisely how measurement reveals these putative underlying (first-order) states. *See generally* Marco Genovese, *Research on Hidden Variable Theories: A Review of Recent Progresses*, 413 PHYSICS REPS. 319 (2005).

[532] Such constraints might be a third-order immunity that prevents any second-order physical process from generally changing the underlying laws. *See supra* Part III.B.2 (describing a quantum formalism for higher-order relations).

[533] Unlike a legal system, the higher-order physical processes need not be "volitional," though there would need to be some fundamental, physical difference between higher- and first-order physical processes. For instance, higher-order physical processes could occur in a separate physical dimension or dimensions ("higher-order space") and depend upon separate —and, very likely, different—sorts of components than the first-order physical processes.

[534] For instance, Mumford and Anjum introduce a new theory of causality based on "powers," but those powers are simply first-order dispositional properties that they model using vectors. *See* STEPHEN MUMFORD & RANI LILL ANJUM, GETTING CAUSES FROM POWERS 22–30 (2011). As such, this approach cannot model second-order causal processes, such as quantum measurement.

[535] *See generally* Richard A. Posner, *The Role of the Judge in the Twenty-First Century*, 86 B.U. L. REV. 1049, 1051 (2006) ("John Roberts at his triumphal confirmation hearing . . . said that the judge . . . is merely an umpire, calling balls and strikes. . . . No serious person thinks

Related, describing quantum measurement as a second-order process also helps cure recurring problems regarding the ontology of spacetime and the nature of fields and matter.[536] For instance, some physicists adhere to the so-called "many worlds" theory, whereby—continuing with the earlier hypothetical—the particle *P* traveling in the external electromagnetic field is viewed as traveling down each potential path from point 1 to point 2 *both before and after measurement*.[537] Yet, as explained, if an observer attempts to measure which path the particle travels along, the observer would find the particle to have travelled down only one of the paths.[538] Rather, the many worlds theory posits that a separate, but unobserved, "universe"—in essence, a slightly altered "copy" of the entire physical universe—is created at the time of measurement for each of the other paths not measured by our observer-of-interest.[539]

The motivation behind this seemingly fantastical theory is that if quantum mechanics truly describes both the state of the measuring instrument and the measured particle—and describes the entire universe-at-large for that matter—then measurement cannot ultimately alter the underlying quantum nature of the universe.[540] That is, each quantum possibility must continue to exist after measurement, just as before measurement.[541] Rather, on this view, what is unique about observation is "simply" that it allows an observer to perceive a particle in one of those possible states, i.e., one of those universes.[542]

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that the rules that judges in our system apply, particularly appellate judges and most particularly the Justices of the U.S. Supreme Court, are given to them the way the rules of baseball are given to umpires. The rules are created by the judges themselves. . . . To decide [an interesting case] the formalist needs a meta-principle . . . .").

536 *See generally* MAX TEGMARK, THE MATHEMATICAL UNIVERSE: MY QUEST FOR THE ULTIMATE NATURE OF REALITY (2008) (positing that the universe is a mathematical structure and describing the nature of fields, matter, spacetime, and measurement in this context).

537 *See generally* DAVID WALLACE, THE EMERGENT MULTIVERSE: QUANTUM THEORY ACCORDING TO THE EVERETT INTERPRETATION (2012) (describing and examining the many-worlds theory in detail).

538 *See generally id.* at 119–20 (discussing the role "branching" in the context of the many-worlds approach). To be certain, the example presented here is somewhat idealized for purposes of exposition.

539 *See generally id.* at ch. 4 (addressing the notion of measurement probability in the context of branching universes).

540 *See generally* Hugh Everett III, *"Relative State" Formulation of Quantum Mechanics*, 29 REV. MOD. PHYSICS 454, 455–59 (1957) (contending that the quantum formalism should apply uniformly to both the measured particle and the measuring instrument, as well as to the entire universe).

541 *See id.* (arguing that all possible outcomes of quantum measurements persist in distinct branches of the "universal" wave function).

542 *See id.* (noting that the observer perceives one of the possible branching measurement states).

From a first-order perspective, the belief that measurement cannot change the underlying quantum nature of the universe because the measurement process is itself quantum "in nature" seems plausible.[543] However, if measurement is a second-order process, then this belief is not only wrong, but simply irrelevant.[544] Specifically, quantum measurement can change first-order quantum states precisely because it is *not* properly described by present-day quantum mechanics, at least completely.[545] On a second-order view, no "many worlds upon measurement" is required—rather, the second-order process of measurement collapses the wave function $|\psi\rangle$ of the particle P (coupled with a set of applicable particles in the measuring device) to result in a record of a particular classical state of P.[546] The measuring process changes first-order relations such that the measured state becomes a pure, classical *duty* relation.[547] Although one might posit that the unmeasured states still exist somewhere in another universe, doing so is entirely unnecessary to explaining a second-order quantum measurement process.[548]

Finally, although a second-order theory of quantum measurement does not in itself answer the ultimate question of what makes one agglomeration of particles a measuring instrument and another not, it

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[543] *See* WALLACE, *supra* note 537, at 22–24.

[544] Again, contrast a judge's first-order obligations not to assault the litigating parties with the judge's second-order power to decide a case. Although there is a "coupling" of the judge to the parties via first-order obligations—and although this coupling cannot "decide" the outcome of the case—this legal fact ultimately becomes irrelevant given the judge's second-order power to determine the ultimate outcome. *See supra* notes 517–21 and accompanying text.

[545] Although it is possible that a first-order process could align the probabilities of particular outcomes of a quantum measurement precisely so that one outcome is at a 100% and all others are at 0%, like in law, it is exceedingly unlikely that a first-order process could reliably and routinely result in such an alignment, particularly in an instantaneous (or at least near-instantaneous) manner. In contrast, a second-order physical process—by definition—would routinely and reliable result in an instantaneous (or near-instantaneous) measurement. *See generally* Angelo Bassi et al., *Models of Wave-Function Collapse, Underlying Theories, and Experimental Tests*, 85 REV. MOD. PHYSICS 471, 483 (2013) (noting the "instantaneous" nature of the collapse of the wave function upon measurement).

[546] Of course, the record of the measured state may deviate from the result of "classical" physical law—the state is "classical" only in the sense that it collapses to a purely classical first-order Hohfeldian duty relation.

[547] Compare the process of legal judgment, which can "collapse" a probabilistic legal "superposition" of Hohfeldian first-order legal states into a single classical state. *See supra* notes 191–207 and accompanying text (explaining the process of judgment in the context of probabilistic legal states).

[548] Similar to a legal judgment, a judge deciding via a second-order power that a litigant previously in a probabilistic state of duty and privilege is now decidedly in a state of duty does not imply that the privilege state somehow persists in an alternate legal universe. Nor in doing so is there any benefit in explaining the judgment process.

provides an essential backdrop to doing so.[549] Returning to the analogous situation of a judge making a judgment, certain critical features emerge.[550] One, recall that a judge acquires relevant second-order (or, in general, *n*th-order) adjudicative powers via the exercise by another legal actor of some *n+1*-th power pursuant to some state of affairs, S_{n+1}.[551] So while legal actors do not ordinarily hold adjudicative *n*th-order powers, when those particular conditions S_{n+1} obtain for a given ordinary actor, that actor will then possess a limited adjudicative power to decide disputes.[552] In the legal world, we know how to specify S_{n+1}—an ordinary legal actor becomes a judge either through a specific appointment process or by popular election.[553]

What are the applicable states of affairs, S_{n+1}, required to provide a physical entity with quantum measurement powers?[554] As a starting point, it is critical to realize that from a first-order perspective, the quantum measuring device is itself a collection of particles that are described by (the first-order relations of) quantum mechanics.[555] If we disaggregate the particles of the measuring device from those "external" to it, then the combined state of the device and the particle to be measured—indeed, the state of the universe of at-large—is one big wave function $|\psi\rangle_{\text{Universe}}$ that

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[549] In this regard, from a second-order perspective, there is nothing special about "measurement." Rather, the same second-order process that results in a measurement—e.g., "spin up" on some particular measuring device—may collapse the wave function in a physical process occurring wholly in the absence of some "measuring device" that records outcomes. The recognition that collapse extends well-beyond observation in the traditional sense helps to solve apparent paradoxes such a Schrodinger's Cat and Wigner's Friend, because the collapse of the wave function need not rely on any external "classical" or "non-quantum" measuring device. *See generally* David Lewis, *How Many Lives Has Schrödinger's Cat?*, *in* MANY WORLDS?: EVERETT, QUANTUM THEORY, & REALITY 3, 3–22 (Simon Saunders et al. eds., 2010) (discussing the Schrodinger's Cat and Wigner's Friend paradoxes).

[550] *See supra* notes 191–207 and accompanying text.

[551] *See supra* notes 149–53 and accompanying text.

[552] *See supra* notes 191–200 and accompanying text.

[553] *See generally* Richard A. Posner, *An Economic Approach to the Law of Evidence*, 51 STAN. L. REV. 1477, 1495–96 (1999) ("American popular mistrust of judges has a further significance. It has resulted in most American judges being elected rather than appointed and in keeping judicial salaries well below the opportunity costs of the ablest lawyers.").

[554] *Cf.* Maximilian Schlosshauer, *Decoherence, the Measurement Problem, and Interpretations of Quantum Mechanics*, 76 REV. MOD. PHYSICS 1267, 1272 (2005) ("When quantum mechanics is applied to an isolated composite object consisting of a system S and an apparatus A, it cannot determine which observable of the system has been measured – in obvious contrast to our experience of the workings of measuring devices that seem to be 'designed' to measure certain quantities.").

[555] *See* NORSEN, *supra* note 236, at 59–69.

exists in a superposition of probability states.[556] More precisely, one cannot posit a measuring device separate from the rest of $|\psi\rangle_{\text{Universe}}$ without a theory of where particular "objects" that constitute a portion of $|\psi\rangle_{\text{Universe}}$ end and other objects start.[557] Fortunately, $|\psi\rangle_{\text{Universe}}$—to the extent an observer is concerned about some locally measurable aspect of it—can be effectively decomposed into separate components. In other words, like the first-order relations in social law, the first-order relations in physical law exhibit modularity.[558] For simplicity of exposition, let one component, $|\psi\rangle_{\text{MP}}$, be the wave function of our putative measuring device M coupled to the particle P it is to measure.[559] What is it about $|\psi\rangle_{\text{MP}}$ that causes the so-called wave function $|\psi\rangle_{\text{P}}$ of P to collapse and an observable property of the particle to be measured?[560]

Here, structural Hohfeldian theory cannot answer the question.[561] However, knowing that whatever this measurement property may turn out to be is triggered by a state of affairs S_{n+1} that results in the exercise of a $(n+1)$th-order power is important for assessing possible answers.[562] For instance, on one theory of measurement, it is human (and perhaps animal) consciousness that collapses the wave function.[563] That is, all objects without consciousness interacting with other objects without consciousness remain in probabilistic superpositions of states, and it is only when one of these objects interacts with a conscious entity that these objects' wave functions can collapse.[564] This approach is usually criticized on the grounds that a conscious observer is subject to the same laws of quantum mechanics

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556 *See generally* James B. Hartle & Stephen W. Hawking, *Wave Function of the Universe*, 28 PHYSICAL REV. D 2960, 2960–62 (1983) ("attempt[ing] to apply quantum mechanics to the Universe as a whole").

557 *See* Zurek, *supra* note 499, at 716 (noting how Bohr's solution draws a "border" between the measuring device and the rest of the quantum world).

558 *See supra* notes 318–35 (discussing legal modularity).

559 *See* Norsen, *supra* note 236, at 59–69.

560 *See id.*

561 By "structural" Hohfeldian theory, I refer to the structural formalism required to express all forms of law—including scientific law—as described earlier. *See supra* notes 413–22 (discussing the application of Hohfeldian formalism to describe scientific law).

562 An analogous situation would be an anthropologist trying to determine how a specific person acquired the ability to decide disputes among ordinary citizens. If the anthropologist used Hohfeldian first-order rights and duties and related first-order actions to explain the decisionmaker's ability, it would be woefully incomplete. Rather, only with an explanation of power-conferring rules and the related states of affairs or actions that implement such rules would the anthropologist be able to adequately explain such authority. *See generally* HART, *supra* note 1, at 27–28 (explaining the notion of power-conferring rules).

563 *See generally* HENRY P. STAPP, MIND, MATTER, AND QUANTUM MECHANICS (A.C. Elitzur et al., eds., 3rd ed. 2009) (arguing that human consciousness collapses the wave function).

564 *See generally id.*

as that observed, so it seems impossible that the quantum mechanical observer is any different from any other collection of particles.[565] The retort of proponents is that somehow consciousness exists outside of the quantum mechanical framework.[566] But, as described above, the "measuring device is quantum mechanical" concern fades away—whether for a conscious entity or for some "ordinary" measuring instrument—with a second-order approach to quantum measurement.[567] In particular, although consciousness could be a sufficient (and perhaps necessary) condition to acquire second- and higher-order measurement powers, it is unclear—at least at first blush—what makes it different from other kinds of potential measuring processes.[568]

It does seem that a second-order approach can rule out "complexity,"[569] at least as a sufficient condition, to make a measuring device. In particular, there is nothing different in kind in Hohfeldian theory about combining many legal actors and relations an undifferentiated agglomeration of legal relations from just a few legal actors related by a few legal relations.[570] Of course, doing so will often have *practical* consequences for legal actors—there is no doubt a huge difference between a modern administrative state and a communal village council—but size alone, i.e., without some additional component whereby size changes the underlying nature of the set of legal relations, does not seem to answer the question.[571]

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[565] *See generally* David Bourget, *Quantum Leaps in Philosophy of Mind: A Critique of Stapp's Theory*, 11 J. CONSCIOUSNESS STUD. 17 (2004).

[566] *See, e.g.*, DAVID J. CHALMERS, THE CONSCIOUS MIND: IN SEARCH OF A FUNDAMENTAL THEORY (1996) (discussing approaches such as panpsychism and property dualism that place consciousness outside the context of quantum mechanical laws).

[567] *See supra* notes 543–49 and accompanying text.

[568] On the other hand, perhaps one essential ingredient of consciousness is the capacity to exercise higher-order powers. In this regard, note that higher-order powers may be viewed as a first step to solving the problem of "free will." In particular, a first-order physical world may be completely deterministic, yet at the same time those holding second- and higher-order powers would have the ability to alter the first-order legal relations so as to affect objects by the exercise of their higher-order "free" will.

[569] By "complexity," I mean the total degrees of freedom of the system under consideration.

[570] This also seems to imply that "size" is not a sufficient determinant—namely, whether an object is macroscopic or microscopic is not the sole cause of whether it may perform quantum measurements. This observation implies that there may be macroscopic objects wholly of a quantum nature. Although building a macroscopic quantum object, e.g., a quantum computer, may be of severe practical difficulty—at least the Hohfeldian theory would not rule it out. On the other hand, size does seem correlated with whatever ultimate set of properties yields measurement powers.

[571] Although the size and complexity of a Hohfeldian system may affect the "modularity" of the system such that certain systems may in practice be considered independent of others, the

On the other hand, as the size and complexity of a physical system increases, the likelihood of "decoherence" of a sub-system of concern increases.[572] In particular, decoherence describes the process whereby a set of a particles in quantum superposition loses its quantum "coherent" state through interaction with some external environment.[573] Although decoherence does not fully answer the measurement question, it may play a fundamental role in converting quantum states to classical measurements.[574] Typically, the missing piece in the decoherence approach to measurement is supplied by the many-worlds theory.[575] Yet, as discussed earlier, by viewing measurement as a higher-order physical process, the ontological need for many branching universes disappears.[576] Perhaps supplementing decoherence with a suitable understanding of how second-order physical processes ultimately collapse the wave function will supply the answer to the measurement problem.

In the meantime, we can speculate that the *why* of what makes a measuring device very likely depends upon certain kinds of spatio-temporal relationships among the underlying particles and related fields making up the relevant aspects of the measuring device, wherein the scope of the "measuring device" possibly includes human or other observers.[577]

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fundamental nature of complex Hohfeldian systems—at least from a legal relations standpoint—is formally the same as simple ones. *See generally* Sichelman & Smith, *supra* note 323.

[572] *See* Schlosshauer, *supra* note 554, at 1273–75 (discussing how size increases the likelihood of decoherence).

[573] *See id.* at 1267–78.

[574] Specifically, the interaction of the measurement device (or any external physical system leading to collapse) will interact with a system to be measured in such way that the combined superposition of measuring device and system may result in "decoherence" that in effect diagonalizes the density matrix of the system so as to result in multiple, potential classical states. *See id.* at 1275–83. Yet, there is no reason why decoherence alone selects one of these classical states during measurement. One possible solution is the so-called many-worlds theory, which further posits that following decoherence, the universe splits into multiple branches, resulting in effective measurement. Besides lacking parsimony, the many-worlds theory is plagued by other notable problems. *See generally* Adler, *supra* note 489 (explaining why decoherence does not solve the measurement problem). The second-order approach to quantum measurement offered here arguably provides an avenue that avoids these concerns.

[575] *See* Schlosshauer, *supra* note 554, at 1288–93.

[576] *See supra* notes 536–49 and accompanying text.

[577] To be certain, viewing measurement as a second-order physical process in no manner necessitates the need for an "external" observer, much less "consciousness," to collapse the wave function. In this regard, it may be the case that there are different second-order processes that may in effect collapse the wave function. Compare the case of second-order processes in law, which although they partake of the same fundamental Hohfeldian powers and related second-order relations, may result from very different actions on the part of the legal actors exercising those powers. *See supra* notes 74–94 and accompanying text (explaining powers, liabilities, immunities, and disabilities).

These relationships will give rise to second- and higher-order powers that will enable the device to collapse wave functions.[578]

Although these higher-order relationships may in part depend upon a certain complexity, may be correlated somehow with consciousness, or may be carried out via some other mechanism (e.g., curved spacetime), ultimately the approach to solving the quantum measurement problem must be to determine exactly which sets of first-order relationships give rise to second-order (and, potentially, higher-order) measurement processes and which do not.[579] In the end, while Hohfeldian structural theory cannot directly answer such questions, it provides a framework that hopefully will enrich the process.[580]

CONCLUSION

Hohfeld's feat of reducing all legal relations to two families of four relations had the profound effect of providing a logical framework to describe the "structure" of the law.[581] Later scholars formalized this framework into formal, deontic logic-oriented models.[582] Yet, these models neither provided descriptions of legal systems using quantitative measures, such as temperature and information entropy, nor did they allow for the possibility of inherently indeterministic legal relations.[583]

This Article provides a mathematical model of the Hohfeldian system that lends itself to a variety of mathematical-physical measures to describe the properties of legal systems in a precise quantitative fashion.[584]

[578] *See supra* notes 507–31 (positing that quantum measurement is a "second-order" physical process). Third- and higher-order processes will play a role in determining which second-order processes collapse the wave function and which processes do not, but the approach offered here identifies the fundamental mechanism behind quantum measurement as a second-order process.

[579] It is possible that all measurement is the result of random second-order measurement processes that do not depend on the specific configuration of first-order processes, but even then, the specific first-order configuration of the system of interest will presumably be related to how measurement affects the state of the system. *Cf.* Giancarlo C. Ghirardi, Alberto Rimini & Tullio Weber, *Unified Dynamics for Microscopic and Macroscopic Systems*, 34 PHYSICAL REV. D 470 (1986) (postulating the "collapse" of the wave function is a random event).

[580] Similar to the synthetic process of applying the Hohfeldian structure to particular legal content to form legal "propositions," understanding quantum measurement will require a synthetic application of the second-order formalism of physical law to the particular content of physics—that is, matter, fields, spacetime, and the like. *See supra* Part V.A (describing the "synthetic" process of forming legal "propositions" as a merger of "structure" and "content").

[581] *See supra* Part I (describing Hohfeld's typology).

[582] *See supra* Part II (describing a logical formalization of the Hohfeldian typology).

[583] *See supra* notes 5 & 40 (discussing the prior literature).

[584] *See supra* Parts III & IV (introducing a mathematical formalization of the Hohfeldian typology and related "physical" measures of the properties of legal systems).

Additionally, the model incorporates legal indeterminacy via quantum mechanical formalism. These results lead to several significant consequences.[585] First, contrary to the views of some post-classical scholars, the inherent indeterminism and the influence of extra-legal factors on adjudication of the law do not impugn an underlying Hohfeldian logic to the law.[586] Although the law is not *classically* rational, its structure is rational within the framework of a Hohfeldian quantum logic.[587] Indeed, in the model proposed here, all legal propositions describing the obligations—and any states of affairs changing, terminating, or creating those obligations—must adhere to this logical structure of the law.[588] This requirement applies not only with the private law sphere, but to all obligations affecting complex legal entities, including "the State."[589]

Second, on a more practical level, the mathematical properties of legal systems and the quantum description of legal relations proposed here offers quantitative measures that can precisely describe the properties of legal systems.[590] For instance, measures such as legal temperature and entropy can be used to model the rate of change and level of indeterminacy of legal relations, which in turn can be used as inputs into quantitative measures of well-known legal concepts such as firm boundaries, information costs, and legal modularity.[591] Because legal relations may be suitably described by tensors and vectors, many other physical and mathematical properties can be incorporated into the quantitative description of legal systems.[592]

Third, the formalism helps shed light on the nature of legal rules and legal artificial intelligence.[593] Specifically, legal propositions are the "synthesis" of Hohfeldian "structure" (legal relations, legal actors, and

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585 *See supra* Part III.B (offering a probabilistic extension of the Hohfeldian legal relations relying upon the formalism of quantum mechanics).

586 *See supra* Part V.A (discussing the nature of legal reasoning in view of the probabilistic framework of the legal relations).

587 *See supra* Part V.A (contrasting the "structure" with the "content" of the law).

588 *See supra* Part V.A (contending that all legal propositions, even probabilistic ones that depend on "standards," must adhere to the Hohfeldian structural framework).

589 *See supra* note 110 and accompanying text (describing the application of the Hohfeldian framework to the State).

590 *See supra* Part IV (describing the entropy and temperature of a legal system).

591 *See id.* (explaining the relationship between legal entropy and temperature and legal modularity).

592 *See id.* (describing the energy of a legal system).

593 *See supra* Part V (discussing the application of the model presented herein to legal reasoning and AI & law).

states of affairs) with specific "content" that fills that structure.[594] Legal rules provide the instructions to construct ("synthesize") legal propositions, which describe the rights and obligations—and the ability to change, create, and terminate those legal entitlements—in specific situations for specific legal actors.[595] In terms of legal AI, legal relations may be viewed as collections of tensor- and vector-based, probabilistic superpositions spanning a multi-dimensional Hohfeldian space.[596] As such, legal relations—including the dynamic changes in and interactions among these relations—can be suitably represented by the "quantum" bits (i.e., qubits) of a quantum computer.[597]

Fourth, the recent field of quantum game theory has shown that by expanding the classical strategy space to allow for "quantum" strategies, players may be able to improve upon their payoffs.[598] Elsewhere, I employ the quantum model of second-order powers presented here to show that the government can increase social welfare by engaging in effective quantum strategies as an intentional "mechanism designer" of indeterministic legal regimes.[599]

Last, it is arguably no coincidence that the propositional structure of legal rules is homologous to the structure of physical laws.[600] On this approach, the structural formalism presented here proves useful in exploring the ontological foundations and origins of other rule-based systems, including scientific laws, such as classical and quantum physics.[601] In particular, the nature of quantum measurement may be better understood as a higher-order physical process embedded within the familiar first-order physical world.[602]

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594 *See supra* Part V.A (explaining the synthetic process of merging legal content into legal structure to generate specific legal propositions).

595 *See id.*

596 *See supra* Part V.B (discussing how legal systems may be modeled as a multidimensional data arrays of qubits).

597 *See id.* (describing how a quantum computer is best positioned to simulate the dynamic evolution of a legal system).

598 *See supra* Part V.C (describing the application of the model presented herein to the development of an "endogenous" quantum game theory).

599 *See* Sichelman, *supra* note 42.

600 *See supra* Part VI (applying the model developed herein to the description of rules generally).

601 *See supra* Part VI.B (exploring the structure of physical laws).

602 *See supra* Part VI.B.2 (asserting that a purely "first-order" approach to quantum mechanics cannot properly describe the measurement process).